Problem 1  Overcomplete expansion in $\mathbb{R}^2$

Consider the following example discussed in class. For the vectors

\[
\begin{align*}
e_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & e_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & e_3 &= e_1 - e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{align*}
\]

we found that any vector $x \in \mathbb{R}^2$ can be represented according to

\[
x = \langle x, \tilde{e}_1 \rangle e_1 + \langle x, \tilde{e}_2 \rangle e_2 + \langle x, \tilde{e}_3 \rangle e_3
\]

where

\[
\tilde{e}_1 = 2e_1, \quad \tilde{e}_2 = -e_3, \quad \tilde{e}_3 = -e_1.
\]

(a) Find another set of vectors $\tilde{e}_1', \tilde{e}_2', \tilde{e}_3'$, neither of which is collinear to neither of the vectors $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ and such that any vector $x \in \mathbb{R}^2$ can be represented as

\[
x = \langle x, \tilde{e}_1' \rangle e_1 + \langle x, \tilde{e}_2' \rangle e_2 + \langle x, \tilde{e}_3' \rangle e_3.
\]

*Hint:* Look for another right-inverse of the matrix

\[
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.
\]

(b) Now consider the following example discussed in class. For the vectors

\[
\begin{align*}
e_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & e_2 &= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, & \tilde{e}_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & \tilde{e}_2 &= \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}
\end{align*}
\]

any vector $x \in \mathbb{R}^2$ can be represented as

\[
x = \langle x, e_1 \rangle \tilde{e}_1 + \langle x, e_2 \rangle \tilde{e}_2.
\]

Show that $x$ can also be written as

\[
x = \langle x, \tilde{e}_1 \rangle e_1 + \langle x, \tilde{e}_2 \rangle e_2.
\]
Is it possible to find two vectors $\mathbf{e}_1', \mathbf{e}_2'$, neither of which is collinear to neither of the vectors $\mathbf{e}_1, \mathbf{e}_2$ such that
\[
\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1' \rangle \tilde{\mathbf{e}}_1 + \langle \mathbf{x}, \mathbf{e}_2' \rangle \tilde{\mathbf{e}}_2.
\]
If the answer is "yes", find these vectors. If the answer is "no", explain why this is not possible.

**Problem 2  Equality in the Cauchy-Schwarz inequality**

Prove that if the elements $x$ and $y$ of a complex Hilbert space $H$ satisfy $|\langle x, y \rangle| = \|x\|\|y\|$ and $y \neq 0$, then $x = cy$ for some $c \in \mathbb{C}$.

**Hint**: Assume $\|x\| = \|y\| = 1$ and $\langle x, y \rangle = 1$. Then $x - y$ and $x$ are orthogonal, while $x = x - y + y$. Therefore, $\|x\|^2 = \|x - y\|^2 + \|y\|^2$.

**Problem 3  Parallelogram law**

a) Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $\| \cdot \|$ the norm induced by $\langle \cdot, \cdot \rangle$. Show that for all $x, y \in X$, the following holds
\[
\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1}
\]

b) Let $(X, \| \cdot \|)$ be a normed space. Show that if (1) holds for all $x, y \in X$, then there exists an inner product $\langle \cdot, \cdot \rangle$ such that $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$. For simplicity, you may assume that $X$ is a real normed space.

Use the last statement to show that $\ell^2(\mathbb{Z})$ is a Hilbert space. What about $\ell^1(\mathbb{Z})$?

**Recall**: $\ell^p(\mathbb{Z})$, $p \in [1, \infty)$, is the space
\[
\ell^p(\mathbb{Z}) \triangleq \left\{ \{u_k\}_{k \in \mathbb{Z}} : \sum_{k=-\infty}^{+\infty} |u_k|^p < \infty \right\}
\]
equipped with the norm
\[
\|u\|_{\ell^p(\mathbb{Z})} \triangleq \left( \sum_{k=-\infty}^{+\infty} |u_k|^p \right)^{1/p}.
\]

**Problem 4  Bessel’s inequality**

Let $H$ be a Hilbert space.

a) Assume that $H$ has dimension $N$ and take a set $\{\mathbf{e}_k\}_{k=1}^M$ of $M \leq N$ orthonormal vectors in $H$. Show that the following holds:
\[
\sum_{k=1}^{M} |\langle x, \mathbf{e}_k \rangle|^2 \leq \|x\|^2.
\]
b) Assume that $\mathcal{H}$ has infinite dimension and let $\{e_k\}_{k=1}^\infty$ be a set of orthonormal vectors of $\mathcal{H}$. Show that the series $\sum_k |\langle x, e_k \rangle|^2$ is convergent and satisfies

$$\sum_{k=1}^\infty |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

for all $x \in \mathcal{H}$.

Problem 5  Characterization of an orthonormal basis in finite dimensions

Let $\mathcal{H}$ be a Hilbert space of dimension $N$ and $\{e_k\}_{k=1}^N$ a collection of vectors of $\mathcal{H}$. Show that

$$\|x\|^2 = \sum_{k=1}^N |\langle x, e_k \rangle|^2$$

for all $x \in \mathcal{H}$ if and only if $\{e_k\}_{k=1}^N$ is an orthonormal basis.

Problem 6  Discrete Fourier Transform (DFT) as a signal expansion

The DFT of an $N$-point signal $f(n)$, $n = 0, 1, \ldots, N-1$, is defined as

$$\hat{f}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{-i2\pi \frac{k}{N} n}$$

Find the corresponding inverse transform and show that the DFT can be interpreted as a signal expansion in $\mathbb{C}^N$. 