Harmonic Analysis: Theory and Applications in Advanced Signal Processing

Spring semester 2015

Homework 2 - Solutions available on March 31, 2015

Problem 1  Frames for $\mathbb{C}^M$
Assume that $\{f_k\}_{k=1}^N$ is a frame for $\mathbb{C}^M$. Prove that the $2N$ vectors consisting of the real parts and of the imaginary parts of the frame vectors constitute a frame for $\mathbb{R}^M$.

Problem 2  Tight frames
Let $\{f_k\}_{k=0}^\infty$ be a frame for the Hilbert space $\mathcal{H}$. Show that the following statements are equivalent:
(i) $\{f_k\}_{k=0}^\infty$ is tight
(ii) $\{f_k\}_{k=0}^\infty$ has a dual of the form $g_k = Cf_k$ for some constant $C > 0$.

Problem 3  Unitary transformation of a frame
Let $\{f_k\}_{k\in\mathcal{K}}$ be a frame for a Hilbert space $\mathcal{H}$ with frame bounds $A$ and $B$. Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. Show that the set $\{Uf_k\}_{k\in\mathcal{K}}$ is again a frame for $\mathcal{H}$ and compute the corresponding frame bounds.

Problem 4  Redundancy of a frame
Let $\{f_k\}_{k=1}^N$ be a frame for $\mathbb{C}^M$ with $N > M$. Assume that the frame elements are normalized such that $\|f_k\| = 1$ for all $k$. The ratio $N/M$ is called redundancy of the frame.

a) Assume that $\{f_k\}_{k=1}^N$ is a tight frame with frame bound $A$. Show that $A = N/M$.

b) Now assume that $A$ and $B$ are lower and upper frame bounds of $\{f_k\}_{k=1}^N$, respectively. Show that $A \leq N/M \leq B$.

Problem 5  Tight frame as an orthogonal projection of an ONB
Let $\{e_k\}_{k=1}^N$ be an orthonormal basis for an $N$-dimensional Hilbert space $\mathcal{H}$. For $M < N$, let $\mathcal{H}'$ be an $M$-dimensional subspace of $\mathcal{H}$. Let $P : \mathcal{H} \to \mathcal{H}'$ be the orthogonal projection onto $\mathcal{H}'$. Show that $\{Pe_k\}_{k=1}^N$ is a tight frame for $\mathcal{H}'$. Find the corresponding frame bound.
Problem 6  Frame bounds

Prove that the upper and lower frame bounds are unrelated: In an arbitrary Hilbert space $\mathcal{H}$ find a set $\{f_k\}_{k \in K}$ with an upper frame bound $B < \infty$ but with the largest lower frame bound $A = 0$; find another sequence $\{f_k\}_{k \in K}$ with a lower frame bound $A > 0$ but with the smallest upper frame bound $B = \infty$. Is it possible to find corresponding examples in the finite-dimensional space $\mathbb{C}^M$?

Problem 7  Weyl-Heisenberg frames in finite dimensions

This is a Matlab exercise. The point of the exercise is to understand what the abstract concept and construct a discrete-time Weyl-Heisenberg set according to parameters $T$, $K$ in Matlab, or discrete samples of the continuous-time Gaussian waveform prototype vector $g$.

Consider the space $\mathbb{C}^M$. Take $M$ to be a large number, such that your signals resemble continuous-time waveforms, but small enough such that your Matlab program works. Take a prototype vector $g = [g[1] \cdots g[M]]^T \in \mathbb{C}^M$. You can choose, for example, the $fir1(.)$ function in Matlab, or discrete samples of the continuous-time Gaussian waveform $e^{-x^2/2}$. Next, fix the shift parameters $T, K \in \mathbb{N}$ such that $L \triangleq M/T \in \mathbb{N}$. Now define

$$g_{k,\ell}[n] \triangleq g[(n-\ell T) \mod M] e^{2\pi i kn/K}, \quad k = 0, \ldots, K-1, \quad \ell = 0, \ldots, L-1, \quad n = 0, \ldots, M-1$$

and construct a discrete-time Weyl-Heisenberg set according to

$$\{ g_{k,\ell} = [g_{k,\ell}[0] \cdots g_{k,\ell}[M-1]]^T \}_{k=0,\ldots,K-1,\ell=0,\ldots,L-1}. $$

a) Show that the analysis operator $T : \mathbb{C}^M \to \mathbb{C}^{KL}$ can be viewed as a $(KL \times M)$-dimensional matrix. Specify this matrix in terms of $g, T, K, \text{ and } M$.

b) Show that the adjoint of the analysis operator $T^* : \mathbb{C}^{KL} \to \mathbb{C}^M$ can be viewed as an $(M \times KL)$-dimensional matrix. Specify this matrix in terms of $g, T, K, \text{ and } M$.

c) Specify the matrix corresponding to the frame operator $S$ in terms of $g, T, K, \text{ and } M$. Call this matrix $S$. Compute and store this $M \times M$ matrix in Matlab.

d) Given the matrix $S$, check, if the Weyl-Heisenberg system $\{g_{k,\ell}\}_{k=0,\ldots,K-1,\ell=0,\ldots,L-1}$ you started from is a frame. Explain, how you can verify this.

e) Prove that for $K = M$ and $T = 1$ and for every prototype vector $g \neq 0$, the set $\{g_{k,\ell}\}_{k=0,\ldots,K-1,\ell=0,\ldots,L-1}$ is a frame for $\mathbb{C}^M$.

f) For the prototype vector $g$ you have chosen, find two pairs of shift parameters $(T_1, K_1)$ and $(T_2, K_2)$ such that $\{g_{k,\ell}\}_{k=0,\ldots,K-1,\ell=0,\ldots,L-1}$ is a frame for $T = T_1$ and $K = K_1$ and is not a frame for $T = T_2$ and $K = K_2$. For the case where $\{g_{k,\ell}\}_{k=0,\ldots,K-1,\ell=0,\ldots,L-1}$ is a frame, compute the frame bounds.

g) Compute the dual prototype vector $\tilde{g} = [\tilde{g}[1] \cdots \tilde{g}[M]]^T = S^{-1}g$. Prove that the dual frame $\{\tilde{g}_{k,\ell}\}_{k=0,\ldots,K-1,\ell=0,\ldots,L-1}$ is given by time-frequency shifts of $\tilde{g}$, i.e.,

$$\{ \tilde{g}_{k,\ell} = [\tilde{g}_{k,\ell}[0] \cdots \tilde{g}_{k,\ell}[M-1]]^T \}_{k=0,\ldots,K-1,\ell=0,\ldots,L-1}$$

with

$$\tilde{g}_{k,\ell}[n] \triangleq \tilde{g}[(n-\ell T) \mod M] e^{2\pi i kn/K}, \quad k = 0, \ldots, K-1, \quad \ell = 0, \ldots, L-1, \quad n = 0, \ldots, M-1.$$