Problem 1  Frame expansion with noise

We have the following:

\[ E \left\{ \| f - f_w \|_2^2 \right\} = E \left\{ \left\| \frac{1}{A} \sum_{j=1}^{M} w_j g_j \right\|_2^2 \right\} = E \left\{ \frac{1}{A^2} \sum_{j=1}^{M} \sum_{\ell=1}^{M} w_j w_\ell \langle g_j, g_\ell \rangle \right\} \]

\[ = \frac{1}{A^2} \sum_{j=1}^{M} \sum_{\ell=1}^{M} E \{ w_j w_\ell \} \langle g_j, g_\ell \rangle = \frac{N_0}{A^2} \sum_{j=1}^{M} \langle g_j \rangle_1^2 \]

\[ = \frac{N_0 M}{A^2} = \frac{N_0 N}{r}. \]

For any Hilbert space of dimension \( N \), the MSE is inversely proportional to the redundancy. Therefore, it is an advantage to formulate algorithms involving frames than bases, which have redundancy \( r = 1 \).

Problem 2  Fat matrix inversion

a) The equation \( \mathbf{A} \mathbf{x} = \mathbf{y} \) has infinitely many solutions if \( \mathbf{y} \) is in the column range space of \( \mathbf{A} \). This is guaranteed when \( \text{rank} \mathbf{A} = S \), i.e., when all rows of \( \mathbf{A} \) are linearly independent (or equivalently, \( S \) columns of \( \mathbf{A} \) are linearly independent).

The equation \( \mathbf{A} \mathbf{x} = \mathbf{y} \) has no solution if \( \mathbf{y} \) is not in the column range space of \( \mathbf{A} \). This can only happen when \( \text{rank} \mathbf{A} < S \), i.e., when \( \mathbf{A} \) has linearly dependent rows (or equivalently, less than \( S \) linearly independent columns).

b) Solving the equation \( \mathbf{A} \mathbf{x} = \mathbf{y} \) under the constraint that \( x_j = 0 \) for \( j \notin S \) amounts to solving the equation \( \mathbf{A} \mathbf{\hat{x}} = \mathbf{y} \), where \( \mathbf{\hat{A}} \) is the \( S \times S \) matrix obtained by removing the \( N - S \) columns of \( \mathbf{A} \) that are indexed by \( \mathbf{S}^c \) and where the unknown \( \mathbf{\hat{x}} \) is an \( S \)-dimensional vector. The equation \( \mathbf{\hat{A}} \mathbf{\hat{x}} = \mathbf{y} \) has exactly one solution if \( \det \mathbf{\hat{A}} \neq 0 \), i.e., if \( \mathbf{\hat{A}} \) has full rank. Therefore, the equation \( \mathbf{A} \mathbf{x} = \mathbf{y} \) has exactly one solution if the columns \( \{\mathbf{a}_j\}_{j \in \mathbf{S}} \) indexed by \( \mathbf{S} \) are linearly independent.
Problem 3  Eigenvalue decomposition of circulant matrices

a) The $k$th entry of the vector $Cf_n$, $n = 0, \ldots, N - 1$, equals

$$
[Cf_n]_k = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} c_k \omega^{nk} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} c_m \omega^{n(k \oplus m)} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} c_m e^{2\pi i n(k \ominus m) / N} \quad (1)
$$

$$
= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} c_m e^{2\pi i n(k-m) / N} = \frac{1}{\sqrt{N}} e^{\pi i n k / N} \sum_{m=0}^{N-1} c_m e^{-2\pi i nm / N} \quad (2)
$$

$$
= \frac{\omega^{nk}}{\sqrt{N}} \lambda_n = \lambda_n [f_n]_k, \quad (3)
$$

where $\ominus$ denotes the subtraction modulo $N$ in (2), i.e., $k \ominus \ell = k - \ell \mod N$, where the second inequality in (2) follows from the fact that $a \mod N = a - \lfloor a / N \rfloor N$ and where we defined

$$
\lambda_n = \sum_{m=0}^{N-1} c_m e^{-2\pi i nm / N}
$$

in (3). Therefore, $Cf_n = \lambda_n f_n$ for all $n = 0, \ldots, N - 1$ and $C$ can be decomposed as

$$
C = F^H \Lambda F,
$$

where $F$ is the $N \times N$ matrix whose $n$th column is $f_n$ and $\Lambda = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{N-1})$.

b) From a), we know that $C_1$ and $C_2$ can be written as

$$
C_1 = F^H \Lambda_1 F \\
C_2 = F^H \Lambda_2 F,
$$

where $\Lambda_1$ and $\Lambda_2$ are diagonal matrices. Since diagonal matrices commute, it holds that

$$
C_1 C_2 = F^H \Lambda_1 F F^H \Lambda_2 F = F^H \Lambda_1 \Lambda_2 F = F^H \Lambda_2 \Lambda_1 F = F^H \Lambda_2 F F^H \Lambda_1 F = C_2 C_1.
$$

Hence, $C_1$ and $C_2$ commute.

Problem 4  Spectrum and eigenvalues

- For all $f \in L^2[0, 1]$, it holds that

$$
\|Tf\|_{L^2[0, 1]}^2 = \int_0^1 |xf(x)|^2 dx = \int_0^1 \frac{x^2}{x^2} |f(x)|^2 dx \leq \int_0^1 |f(x)|^2 dx = \|f\|_{L^2[0, 1]}^2,
$$

which shows that $T$ is bounded.

- For all $f, g \in L^2[0, 1]$, we have

$$
\langle Tf, g \rangle_{L^2[0, 1]} = \int_0^1 T f(x) g(x) dx = \int_0^1 f(x) \overline{g(x)} dx = \int_0^1 f(x) \overline{g(x)} dx = \int_0^1 f(x) \bar{g(x)} dx = \langle f, Tg \rangle_{L^2[0, 1]}.
$$

Hence, $T^* = T$, which means that $T$ is self-adjoint.
• Assume that there exists \( \lambda \in \mathbb{C} \) and \( f \in L^2[0, 1] \setminus \{0\} \) such that \( T f = \lambda f \). Then

\[
T f(x) = x f(x) = \lambda f(x)
\]

for all \( x \in [0, 1] \). If \( \lambda \notin [0, 1] \), then (4) implies that \( f(x) = 0 \) for all \( x \in [0, 1] \). If \( \lambda \in [0, 1] \), then (4) implies that \( f(x) = 0 \) for all \( x \in [0, 1] \setminus \{\lambda\} \). In both cases, \( f \) is zero almost everywhere, that is, \( f = 0 \) in \( L^2[0, 1] \). This constitutes a contradiction, and therefore, \( T \) has no eigenvalues.

• If \( \lambda \notin [0, 1] \), then \( T - \lambda I \) has an inverse, which is the multiplication by \( 1/(x - \lambda) \), i.e.,

\[
\forall x \in [0, 1], \quad \left((T - \lambda I)^{-1} f\right)(x) = \frac{f(x)}{x - \lambda}.
\]

If \( \lambda \in [0, 1] \), \( T - \lambda I \) is injective (as previously shown) but not surjective. Indeed, \( 1 \notin \mathcal{R}(T - \lambda I) \), because \( x \mapsto 1/(x - \lambda) \) does not belong to \( L^2[0, 1] \). Hence, \( T - \lambda I \) is not invertible, and therefore, the spectrum of \( T \) is \( \text{Sp} \ T = [0, 1] \).

**Problem 5**  
**Haar wavelet expansion**

By definition, we have that \( \text{supp} \Psi = [0, 1) \) and hence, for all \( j, k \in \mathbb{Z} \), it holds that

\[
\text{supp} \Psi_{j,k} = [2^j k, 2^j (k + 1)).
\]

Let \( (j, k) \neq (j', k') \).

• If \( j = j' \) and \( k \neq k' \), then by (5), the support of \( \Psi_{j,k} \) and \( \Psi_{j',k'} \) are disjoint, i.e.,

\[
\text{supp} \Psi_{j,k} \cap \text{supp} \Psi_{j',k'} = \emptyset,
\]

implying that \( \langle \Psi_{j,k}, \Psi_{j',k'} \rangle = 0 \).

• If \( j \neq j' \), assume without loss of generality that \( j > j' \). Then, \( \Psi_{j,k} \) is constant on the support of \( \Psi_{j',k'} \), and since

\[
\int_{-\infty}^{+\infty} \Psi_{j,k}(x) dx = \int_{2^j k}^{2^j (k + 1)} 2^{-j/2} \Psi(2^{-j} x - k) dx = 2^{j/2} \int_{0}^{1} \Psi(x) dx = 0,
\]

it follows that \( \langle \Psi_{j,k}, \Psi_{j',k'} \rangle = 0 \).

This shows that \( \{\Psi_{j,k}\}_{j,k \in \mathbb{Z}} \) forms an orthogonal system. Moreover, we have for all \( j, k \in \mathbb{Z} \),

\[
\|\Psi_{j,k}\|^2 = \int_{-\infty}^{+\infty} |\Psi_{j,k}(x)|^2 dx = \int_{2^j k}^{2^j (k + 1)} 2^{-j/2} |\Psi(2^{-j} x - k)|^2 dx = \int_{0}^{1} |\Psi(x)| dx = 1.
\]

Therefore, \( \{\Psi_{j,k}\}_{j,k \in \mathbb{Z}} \) forms an orthonormal system. As stated in the problem statement, the family \( \{\Psi_{j,k}\}_{j,k \in \mathbb{Z}} \) is in addition complete, and hence forms an orthonormal basis for \( L^2(\mathbb{R}) \). Since the function \( f \) defined in the problem statement belongs to \( L^2(R) \), we can write

\[
\forall x \in \mathbb{R}, \quad f(x) = e^{-|x|} = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \langle f, \Psi_{j,k} \rangle \Psi_{j,k}(x),
\]
where the series coefficients are computed as follows:

\[
\langle f, \Psi_{j,k} \rangle = \int_{-\infty}^{+\infty} f(x) \Psi_{j,k}(x) \, dx = 2^{-j/2} \int_{-\infty}^{+\infty} e^{-|x|} \Psi(2^{-j} x - k) \, dx \\
= 2^{-j/2} \int_{-2^{-j}k}^{2^{-j}(k+1)} e^{-|x|} \Psi(2^{-j} x) \, dx \\
= 2^{-j/2} \int_{-2^{-j}k}^{2^{-j}(k+1)} e^{-|x|} \, dx - 2^{-j/2} \int_{2^{-j}k}^{2^{-j}(k+1)} e^{-|x|} \, dx.
\]

If \( k \geq 0 \), it gives

\[
\langle f, \Psi_{j,k} \rangle = 2^{-j/2} \int_{-2^{-j}k}^{2^{-j}(k+1)} e^{-x} \, dx - 2^{-j/2} \int_{2^{-j}k}^{2^{-j}(k+1)} e^{-x} \, dx = 2^j/2 e^{-2^j k} + 2^j/2 e^{-2^j (k+1)} - 2^{(2-j)/2} e^{-2^j k - 2^j}.
\]

If \( k < 0 \), we have

\[
\langle f, \Psi_{j,k} \rangle = 2^{-j/2} \int_{2^{-j}k}^{2^{-j}(k+1)} e^{x} \, dx - 2^{-j/2} \int_{2^{-j}k}^{2^{-j}(k+1)} e^{x} \, dx = 2^{(2-j)/2} e^{2^j k + 2^j} - 2^{j/2} e^{2^j k} - 2^{j/2} e^{2^j (k+1)}.
\]