Problem 1  Frames

Let $f \in \mathcal{H}$. By assumption, it holds that

$$A \| f \|^2 \leq \sum_{k=0}^{\infty} |\langle f, f_k \rangle|^2 \leq B \| f \|^2$$

(1)

and

$$\sum_{k=0}^{\infty} \| f_k - g_k \|^2 < A.$$  

(2)

Since the inequality in (2) is strict, we can write that there exists $\alpha < 1$ such that

$$\sum_{k=0}^{\infty} \| f_k - g_k \|^2 \leq \alpha A.$$  

(3)

For all $k \in \mathbb{N}$, we have $\langle f, g_k \rangle = \langle f, f_k \rangle - \langle f, f_k - g_k \rangle$, and therefore, applying the triangle inequality $\| u - v \|_{\ell^2(\mathbb{N})} \leq \| u \|_{\ell^2(\mathbb{N})} + \| v \|_{\ell^2(\mathbb{N})}$ in $\ell^2(\mathbb{N})$ to the sequences $u = \{ \langle f, f_k \rangle \}_{k=0}^{\infty}$ and $v = \{ \langle f, f_k - g_k \rangle \}_{k=0}^{\infty}$ gives

$$\left( \sum_{k=0}^{\infty} |\langle f, g_k \rangle|^2 \right)^{1/2} \leq \left( \sum_{k=0}^{\infty} |\langle f, f_k \rangle|^2 \right)^{1/2} + \left( \sum_{k=0}^{\infty} |\langle f, f_k - g_k \rangle|^2 \right)^{1/2}.$$  

(4)

It follows from the Cauchy Schwarz inequality and from (3) that

$$\sum_{k=0}^{\infty} |\langle f, f_k - g_k \rangle|^2 \leq \| f \|^2 \sum_{k=0}^{\infty} \| f_k - g_k \|^2 \leq \alpha A \| f \|^2.$$  

(5)

Combining (4) and (5) and using the right-hand side of (1) yields

$$\left( \sum_{k=0}^{\infty} |\langle f, g_k \rangle|^2 \right)^{1/2} \leq \sqrt{B} \| f \| + \sqrt{\alpha A} \| f \| = (\sqrt{B} + \sqrt{\alpha A}) \| f \|,$$
which establishes the frame upper condition. Likewise, applying the reverse triangle inequality
\[ \|u - v\|_{\mathbb{L}^2(\mathbb{N})} \geq \|u\|_{\mathbb{L}^2(\mathbb{N})} - \|v\|_{\mathbb{L}^2(\mathbb{N})} \]
in \( \ell^2(\mathbb{N}) \), we obtain
\[
\left( \sum_{k=0}^{\infty} |\langle f, g_k \rangle|^2 \right)^{1/2} \geq \left( \sum_{k=0}^{\infty} |\langle f, f_k \rangle|^2 \right)^{1/2} - \left( \sum_{k=0}^{\infty} |\langle f, f_k - g_k \rangle|^2 \right)^{1/2} \\
\geq \sqrt{A} \| f \| - \sqrt{\alpha A} \| f \| = \sqrt{A}(1 - \sqrt{\alpha}) \| f \|,
\]
where used again (5) and the left-hand side of (1). This establishes the frame lower condition
since \( \sqrt{A}(1 - \sqrt{\alpha}) > 0 \) given that \( \alpha < 1 \). Therefore, \( \{g_k\}_{k=0}^{\infty} \) is a frame with bounds \( A(1 - \sqrt{\alpha})^2 \) and \( (\sqrt{B} + \sqrt{\alpha A})^2 \).

**Problem 2**  
**Multiresolution analysis: orthogonal projections**

Let \( f \in \mathcal{H} \). Since \( \mathcal{V}_{j+1} \) is a closed subspace of \( \mathcal{H} \), we have that \( \mathcal{H} = \mathcal{V}_{j+1} \oplus \mathcal{V}_{j+1}^\perp \), implying that \( f \) can be decomposed as
\[ f = v_{j+1} + g, \]
where \( v_{j+1} \in \mathcal{V}_j \) and \( g \in \mathcal{V}_{j+1}^\perp \). Since \( \mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j \), the vector \( v_{j+1} \) can be written as
\[ v_{j+1} = v_j + w_j, \]
where \( v_j \in \mathcal{V}_j \) and \( w_j \in \mathcal{W}_j \).

By definition, we have that \( P_{j+1}f = v_{j+1} \), \( P_jv_{j+1} = v_j \), and \( Q_jv_{j+1} = w_j \). Since \( g \in \mathcal{V}_{j+1}^\perp \), \( g \) is orthogonal to all vectors of \( \mathcal{V}_j \subseteq \mathcal{V}_{j+1} \), and to all vectors of \( \mathcal{W}_j \subseteq \mathcal{V}_{j+1} \). Therefore, we have that
\[
P_{j+1}f = P_jv_{j+1} + P_jg = P_jv_{j+1} = v_j \\
Q_jf = Q_jv_{j+1} + Q_jg = Q_jv_{j+1} = w_j,
\]
which means that
\[ P_{j+1}f = P_jf + Q_jf, \]
for all \( f \in \mathcal{H} \).

**Problem 3**  
**Multiresolution analysis: frequency band-limited functions**

- Let us start by showing that \( \mathcal{V}_j \) is a closed linear subspace of \( L^2(\mathbb{R}) \) for every \( j \in \mathbb{Z} \).

Let \( j \in \mathbb{Z} \). The zero function clearly belongs to \( \mathcal{V}_j \). Moreover, if \( f, g \in \mathcal{V}_j \) and \( \alpha, \beta \in \mathbb{R} \), then \( \text{supp} \hat{f} \subseteq [2^{-j-1}, 2^{-j}] \) and \( \text{supp} \hat{g} \subseteq [2^{-j-1}, 2^{-j}] \), and therefore \( \text{supp}(\alpha \hat{f} + \beta \hat{g}) \subseteq [2^{-j-1}, 2^{-j}] \), implying that \( \alpha f + \beta g \in \mathcal{V}_j \). Hence, \( \mathcal{V}_j \) is a subspace of \( L^2(\mathbb{R}) \).

Now let us take a sequence \( \{f_\ell\}_{\ell \in \mathbb{N}} \) of functions in \( \mathcal{V}_j \) converging in \( L^2(\mathbb{R}) \) to the function \( f \), i.e.,
\[ \lim_{\ell \to +\infty} \|f - f_\ell\|_{L^2(\mathbb{R})} = 0. \]

According to Parseval’s equality, it holds that \( \|f - f_\ell\|_{L^2(\mathbb{R})} = \|\hat{f} - \hat{f}_\ell\|_{L^2(\mathbb{R})} \) for all \( \ell \in \mathbb{N} \). Consequently, we have
\[ \lim_{\ell \to +\infty} \|\hat{f} - \hat{f}_\ell\|_{L^2(\mathbb{R})} = 0. \]
In addition, since \( f_\ell \in \mathcal{V}_j \) for all \( \ell \in \mathbb{N} \), we have for all \( \ell \in \mathbb{N} \),

\[
\|\hat{f} - \hat{f}_\ell\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |\hat{f}(\nu) - \hat{f}_\ell(\nu)|^2 d\nu
\]

\[
= \int_{-2^j}^{2^j-1} |\hat{f}(\nu) - \hat{f}_\ell(\nu)|^2 d\nu + \int_{\mathbb{R}\setminus[-2^j, 2^j-1]} |\hat{f}(\nu) - \hat{f}_\ell(\nu)|^2 d\nu
\]

\[
= \int_{-2^j}^{2^j-1} |\hat{f}(\nu) - \hat{f}_\ell(\nu)|^2 d\nu + \int_{\mathbb{R}\setminus[-2^j, 2^j-1]} |\hat{f}(\nu)|^2 d\nu,
\]

since \( \text{supp} \ f_\ell \subseteq [-2^{-j-1}, 2^{-j-1}] \). Letting \( \ell \to \infty \), we obtain that

\[
\int_{\mathbb{R}\subseteq[-2^j, 2^j-1]} |\hat{f}(\nu)|^2 d\nu = 0,
\]

since \( \lim_{\ell \to +\infty} \|\hat{f} - \hat{f}_\ell\|_{L^2(\mathbb{R})} = 0 \). Therefore, \( \text{supp} \ \hat{f} \subseteq [-2^j, 2^j] \), and hence \( f \in \mathcal{V}_j \).

- For all \( j \in \mathbb{Z} \), if \( f \in \mathcal{V}_j \), then \( \text{supp} \ \hat{f} \subseteq [-2^{-j-1}, 2^{-j-1}] \subseteq [-2^j, 2^j] \), and thus \( f \in \mathcal{V}_{j+1} \). So it holds that

\[
\ldots \subseteq \mathcal{V}_{-2} \subseteq \mathcal{V}_{-1} \subseteq \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \ldots .
\]

- We have

\[
\lim_{j \to -\infty} \mathcal{V}_j = \left\{ f \in L^2(\mathbb{R}) : \text{supp} \ \hat{f} = \{0\} \right\} = \{0\}.
\]

- Likewise, it holds that

\[
\lim_{j \to +\infty} \mathcal{V}_j = \left\{ f \in L^2(\mathbb{R}) : \text{supp} \ \hat{f} = \mathbb{R} \right\} = L^2(\mathbb{R}).
\]

- Let \( j \in \mathbb{Z} \) and \( f \in L^2(\mathbb{R}) \). The Fourier transform of the function \( g \in L^2(\mathbb{R}) \), defined as

\[
\forall t \in \mathbb{R}, \quad g(t) = f(2t),
\]

satisfies

\[
\forall \nu \in \mathbb{R}, \quad \hat{g}(\nu) = \frac{1}{2} \hat{f} \left( \frac{\nu}{2} \right).
\]

If \( f \in \mathcal{V}_j \), it holds that \( \text{supp} \ \hat{f} \subseteq [-2^{j-1}, 2^{j-1}] \). It then follows from (6) that \( \text{supp} \ \hat{g} \subseteq [-2^j, 2^j] \), and hence, \( g \in \mathcal{V}_{j+1} \). Conversely, if \( g \in \mathcal{V}_{j+1} \), \( \text{supp} \ \hat{g} \subseteq [-2^j, 2^j] \), then \( \text{supp} \ \hat{f} \subseteq [-2^{j-1}, 2^{j-1}] \) and hence, \( f \in \mathcal{V}_j \).

- Let \( f \in L^2(\mathbb{R}) \) and \( k \in \mathbb{Z} \). The Fourier transform of the function \( g \in L^2(\mathbb{R}) \), defined as

\[
\forall t \in \mathbb{R}, \quad g(t) = f(t - k),
\]

satisfies

\[
\forall \nu \in \mathbb{R}, \quad \hat{g}(\nu) = e^{-2i\pi k \nu} \hat{f}(\nu).
\]

This shows that \( \text{supp} \ \hat{f} = \text{supp} \ \hat{g} \). Therefore, \( \text{supp} \ \hat{f} \subseteq [-1/2, 1/2] \) whenever \( \text{supp} \ \hat{g} \subseteq [-1/2, 1/2] \), and thus, \( f \in \mathcal{V}_0 \iff g \in \mathcal{V}_0 \).

- Let us choose the scaling function

\[
\forall t \in \mathbb{R}, \quad \phi(t) = \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}.
\]
From the sampling theorem, we know that \( \{\phi(t - k)\}_{k \in \mathbb{Z}} \) forms an orthonormal basis for \( \mathcal{V}_0 \).

We have shown that \( \{\mathcal{V}_j\}_{j \in \mathbb{Z}} \) forms a multiresolution analysis of \( L^2(\mathbb{R}) \). The approximation of a function \( f \in L^2(\mathbb{R}) \) on the scale \( 2^{-j} \) is given by its orthogonal projection onto \( \mathcal{V}_j \),

\[
P_{\mathcal{V}_j}f = \arg \min_{g \in \mathcal{V}_j} \|f - g\|_{L^2(\mathbb{R})}.
\]

From the sampling theorem, we know that its Fourier transform is obtained with a frequency filtering:

\[
\forall \nu \in \mathbb{R}, \quad \widehat{P_{\mathcal{V}_j}f}(\nu) = \begin{cases} \hat{f}(\nu), & \nu \in [-2^{-j}, 2^{-j}] \\ 0, & \text{otherwise} \end{cases}
\]

**Problem 4  Weyl-Heisenberg frames**

As suggested in the problem statement, we start by observing that \( T_{\ell/2}M_k \phi = M_{-\ell/2}T_k \phi \). Indeed, we have that

\[
\forall \nu \in \mathbb{R}, \quad T_{\ell/2}M_k \phi(\nu) = \int_{-\infty}^{+\infty} T_{\ell/2}M_k \phi(t)e^{-2i\pi \nu t} dt
\]

\[
= \int_{-\infty}^{+\infty} e^{2i\pi k(t-\ell/2)} \phi(t-\ell/2)e^{-2i\pi \nu t} dt
\]

\[
= \int_{-\infty}^{+\infty} e^{2i\pi \nu \ell/2} \phi(t)e^{-2i\pi \nu (t+\ell/2)} dt
\]

\[
= e^{-2i\pi \nu \ell/2} \int_{-\infty}^{+\infty} \phi(t)e^{-2i\pi (\nu+k)t} dt
\]

\[
= M_{-\ell/2}T_k \phi(\nu).
\]

Using Parseval’s equality, we can then write that

\[
\sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \left| \left< f, T_{\ell/2}M_k \psi \right>_{L^2(\mathbb{R})} \right|^2 = \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \left| \left< f, \frac{1}{\sqrt{2}} T_{\ell/2}M_k \phi \right>_{L^2(\mathbb{R})} \right|^2
\]

\[
= \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \left| \left< \hat{f}, \frac{1}{\sqrt{2}} T_{\ell/2}M_k \phi \right>_{L^2(\mathbb{R})} \right|^2
\]

\[
= \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \left| \int_{-\infty}^{+\infty} \hat{f}(\nu) \frac{1}{\sqrt{2}} e^{2i\pi \nu \ell/2} T_k \phi(\nu) d\nu \right|^2
\]

Since \( \text{supp} \, \hat{\phi} \subseteq [-1, 1] \), we have that \( \text{supp} \, T_k \hat{\phi} \subseteq [k-1, k+1] \), which gives

\[
\sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \left| \left< f, T_{\ell/2}M_k \psi \right>_{L^2(\mathbb{R})} \right|^2 = \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \left| \int_{-\infty}^{+\infty} \hat{f}(\nu) \frac{1}{\sqrt{2}} e^{-i\pi \nu \ell/2} T_k \phi(\nu) d\nu \right|^2,
\]
where \( e_{\ell/2}(\nu) = e^{2i\pi(\ell/2)\nu} \) for all \( \nu \in [k - 1, k + 1] \). We recognize the standard inner product of \( L^2[k-1,k+1] \):

\[
\sum_{k \in \mathbb{Z}} \left| \sum_{\ell \in \mathbb{Z}} \left< f, T_{\ell/2}^k \Psi \right> \right|^2 = \sum_{k \in \mathbb{Z}} \left< \tilde{f} T_{k} \varphi, \frac{1}{\sqrt{2}} e^{i\pi \ell/2} \right>_{L^2[0,2\pi]}^2.
\]

Since \( \{e_{-\ell/2}/\sqrt{2}\}_{\ell \in \mathbb{Z}} \) forms an orthonormal basis for \( L^2[k-1,k+1] \) for all \( k \in \mathbb{Z} \), we can use again Parseval’s equality to write that

\[
\sum_{k \in \mathbb{Z}} \left| \sum_{\ell \in \mathbb{Z}} \left< f, T_{\ell/2}^k \Psi \right> \right|^2 = \sum_{k \in \mathbb{Z}} \left< \tilde{f} T_{k} \varphi, \frac{1}{\sqrt{2}} e^{i\pi \ell/2} \right>_{L^2[0,2\pi]}^2 = \sum_{k \in \mathbb{Z}} \left< \tilde{f} T_{k} \varphi, \frac{1}{\sqrt{2}} \right>_{L^2[0,2\pi]}^2.
\]

Note that we can exchange the order of the summation and the integral in (8), since the series \( \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} |\tilde{f}(\nu)|^2 |\hat{\varphi}(\nu - k)|^2 \, d\nu \) converges. The equality in (9) shows that \( \{T_{\ell/2}M_k \Psi\}_{k,\ell \in \mathbb{Z}} \) forms a tight frame with frame bound \( A = 1 \).