Problem 1  Image compression using the Haar wavelet basis

a) Write a function in Matlab, which takes as input a discrete signal $x \in \mathbb{C}^N$ of size $N = 2^J$ (e.g., $J = 8$) and computes the Haar wavelet decomposition of $x$ into $[a_0^T \ w_0^T \ w_1^T \ldots \ w_{J-1}^T]^T$, where $w_j$, $j = 0, 1, \ldots, J - 1$, is a vector containing the wavelet coefficients at the scale $2^{-j}$ and $a_0$ contains the approximation coefficients.

Recall: The approximation coefficients $a_j[k]$ and the wavelet coefficients $w_j[k]$ can be computed recursively by applying the following downsampling step for $j = J, J - 1, \ldots, 1$:

$$a_{j-1}[k] = \frac{1}{\sqrt{2}}(a_j[2k] + a_j[2k + 1])$$
$$w_{j-1}[k] = \frac{1}{\sqrt{2}}(a_j[2k] - a_j[2k + 1]),$$

where $k = 0, 1, \ldots, 2^{j-1} - 1$, starting from $a_J[k] = x[k]$ for all $k = 0, 1, \ldots, 2^J - 1$.

b) Write a function in Matlab to reconstruct a signal from its Haar wavelet decomposition $[a_0^T \ w_0^T \ w_1^T \ldots \ w_{J-1}^T]^T$.

c) The Haar wavelet decomposition of an image can be done by performing a downsampling step as described in 1. in the vertical direction, followed by a downsampling step in the horizontal direction.

Write a Matlab function to compute the Haar wavelet decomposition of an image of size $N \times N$ with $N = 2^J$. Then write a Matlab function to reconstruct the original image from this decomposition. As an application, take an image and compute its Haar wavelet decomposition. Plot the magnitude of the coefficients sorted in decreasing order. Now set the 90% smallest coefficients (in magnitude) to zero and apply the reconstruction procedure. How good is the approximation?

Problem 2  Wavelet frame

Let $\varphi, \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be functions whose Fourier transforms satisfy the following properties:

a) $\text{supp}\ \hat{\varphi} \subseteq [-1, 1]$ and $\text{supp}\ \hat{\psi} \subseteq [-2, -1/2] \cup [1/2, 2]$
b) \(|\hat{\varphi}(\nu)|^2 + \sum_{j=0}^{+\infty} |\hat{\psi}(2^{-j}\nu)|^2 = 1\) for all \(\nu \in \mathbb{R}\).

Define for \(j \in \mathbb{N}\) and \(k \in \mathbb{Z}\)
\[
\forall t \in \mathbb{R}, \quad \psi_{-1,k}(t) = \varphi(t - k/2)/\sqrt{2}
\]
\[
\forall t \in \mathbb{R}, \quad \psi_{j,k}(t) = 2^{j/2-1}\psi(2^j t - k/4).
\]

Show that \(\{\psi_{j,k}\}_{j \geq -1, k \in \mathbb{Z}}\) forms a tight frame for \(L^2(\mathbb{R})\) with frame bound 1.

Hint: See solution to Problem 4 of Homework 4.

**Problem 3**  
“\(\ell_0\)-norm”

In the lecture, we defined \(\|x\|_0\) to be the number of nonzero entries in the vector \(x \in \mathbb{C}^N\). Show that \(\|\cdot\|_0\) is not a norm for \(\mathbb{C}^N\), despite the fact that it is often referred to as “\(\ell_0\)-norm”.

**Problem 4**  
Gram matrix

Let \(\{a_k\}_{k=1}^N\) be a set of vectors of \(\mathbb{C}^M\). Show that the Gram matrix \(\{\langle a_k, a_\ell \rangle\}_{k,\ell=1}^N\) is a Hermitian positive-semidefinite matrix, and that it is positive-definite whenever \(\{a_k\}_{k=1}^N\) forms a linearly independent set of vectors.

**Problem 5**  
Compressed sensing

Let \(x \in \mathbb{R}^N\) be a piecewise constant vector with only a small number \(s\) of jumps. That is,
\[
x = \begin{bmatrix}
\alpha_1 & \alpha_1 & \ldots & \alpha_1 \\
\text{block 1} & \alpha_2 & \alpha_2 & \ldots & \alpha_2 \\
& \alpha_3 & \ldots & \alpha_s & \alpha_s \\
& & \ldots & \ldots & \ldots \\
& & & & \alpha_s \\
\end{bmatrix}^T.
\]

Suppose that \(x\) is unknown to us and that we only know the measurement vector
\[
y = Ax \in \mathbb{R}^M,
\]
where \(A \in \mathbb{R}^{M \times N}\) is a known matrix modeling a linear measurement process. In the case where \(M \leq N\), (1) forms an underdetermined system of equations. However, compressed sensing theory tells us that it is all the same possible to recover \(x\) under certain conditions.

Explain how you can recover the vector \(x\) from the knowledge of \(y\) and \(A\) only.