Fundamentals of Wireless Communication

Homework 1
Handout date: March 18, 2014

Problem 1 Identification of LTV Systems

The goal of this problem is to show formally that systems whose spreading function is supported on a rectangle of area one in the Delay-Doppler space are identifiable. The main steps of the proof to be derived follow the proof given in class.

We consider systems, i.e., operators, that can be represented as

\[
(Hx)(t) = \int_\tau h(t, \tau)x(t - \tau) d\tau = \int_\tau \int_\nu S_H(\tau, \nu)x(t - \tau)e^{j2\pi \nu t} d\nu d\tau.
\] (1.1)

We denote the linear space of operators that can be represented according to (1.1) by \( \mathcal{H} \), and define the inner product on this space by

\[
\langle H_1, H_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} S_{H_1}(\tau, \nu)S_{H_2}^*(\tau, \nu)d(\tau, \nu), \quad H_1, H_2 \in \mathcal{H}
\]

with the induced norm \( \|H\|_{\mathcal{H}} = \sqrt{\langle H, H \rangle_{\mathcal{H}}} \). We will consider the set of operators \( Q \subset \mathcal{H} \), defined as

\[
Q = \{ H \in \mathcal{H} : S_H(\tau, \nu) = 0, (\tau, \nu) \not\in \Theta \}
\]

where \( \Theta \) is the rectangle \( \Theta = [0, T) \times [0, 1/T), T \in \mathbb{R} \). Let us next formally define the notion of identifiability. The set \( Q \) is said to be identifiable from an input-output measurement, if there exists a probing signal \( x \) such that for each \( H \in Q \), the action of the operator on the probing signal, \( Hx \), uniquely determines \( H \), i.e., if there exists an \( x \) such that

\[
H_1x = H_2x \implies H_1 = H_2, \quad \forall H_1, H_2 \in Q.
\] (1.2)

Identifiability is hence equivalent to invertibility of the mapping

\[
H \mapsto Hx
\] (1.3)

induced by the probing signal \( x \). Invertibility alone is typically not sufficient as one wants to recover \( H \) from \( Hx \) in a numerically stable fashion, i.e., we want small errors in \( Hx \) to result in small errors in the identified operator. This requirement implies that the inverse of the mapping (1.3) must be continuous (and hence bounded), which finally motivates the following definition.

**Definition 1** We say that \( x \) stably identifies \( Q \) if there exist constants \( 0 < \alpha \leq \beta < \infty \) such that for all \( H \in Q \),

\[
\alpha \|H\|_{\mathcal{H}} \leq \|Hx\| \leq \beta \|H\|_{\mathcal{H}}.
\] (1.4)
1. Show that stable identifiability implies identifiability, i.e., if \( x \) stably identifies \( Q \), then \( x \) satisfies (1.2).

2. Show that
\[
x(t) = \sqrt{T} \sum_{k \in \mathbb{Z}} \delta(t + kT)
\]
stably identifies \( Q \), and determine the constants \( \alpha, \beta \) in the definition of stable identifiability.

*Hint:* Conceptually, the proof is equivalent to that given in class. Start by computing the Zak transform of the response of \( H \in Q \) to \( x \). The Zak transform (with parameter \( T \)) of the signal \( y(t) \) is given by
\[
Z_y(t, f) = \sqrt{T} \sum_{m \in \mathbb{Z}} y(t + mT)e^{-j2\pi mTf}
\]
for \((t, f) \in \Theta\). The Zak transform is a unitary mapping from \( L_2(\mathbb{R}) \) onto \( L_2(\Theta) \). Hence the Zak transform is energy preserving, i.e.,
\[
\int_{\Theta} |Z_y(t, f)|^2 d(t, f) = ||y||^2.
\]
You will also need the Poisson summation formula, i.e., for a signal \( y(t) \) with Fourier transform \( Y(f) \),
\[
\sum_{m \in \mathbb{Z}} Y(f + m/T) = T \sum_{m \in \mathbb{Z}} y(mT)e^{-j2\pi mTf}.
\]

3. Provide an explicit reconstruction formula for the spreading function \( S_H \) of an operator \( H \in Q \) as a function of \( y(t) = (Hx)(t) \).

**Problem 2  Conditional Expectation and Variance**

The conditional expectation of the real-valued continuous random variable \( X \) given the real-valued continuous random variable \( Y \) is commonly denoted as \( \mathbb{E}_X[X \mid Y] \). Although conditional expectation sounds like a number, it is actually a random variable.

1. Assume that \( Y \) has probability density function \( f_Y(y) \) and that \( X \) has conditional probability density function \( f_{X \mid Y}(x \mid y) \) for any \( y \) such that \( f_Y(y) > 0 \). Show that
\[
\mathbb{E}_Y[\mathbb{E}_X[X \mid Y]] = \mathbb{E}_X[X].
\]

2. The conditional variance of the real-valued random variable \( X \), given the real-valued random variable \( Y \), is defined by
\[
\text{Var}[X \mid Y] = \mathbb{E}_X[(X - \mathbb{E}_X[X \mid Y])^2 \mid Y].
\]
Show that
\[
\text{Var}[X] = \mathbb{E}_Y[\text{Var}[X \mid Y]] + \text{Var}[\mathbb{E}_X[X \mid Y]].
\]
Problem 3  Complex Random Vectors

Let $\mathbf{x}$ be an $N$-dimensional zero-mean complex random vector.

1. Let $\hat{\mathbf{x}}$ be a $2N$-dimensional zero-mean real random vector defined as

   \[
   \hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_R \\ \mathbf{x}_I \end{bmatrix}
   \]

   where $\mathbf{x}_R$ and $\mathbf{x}_I$ denote the real and imaginary part of the complex vector $\mathbf{x}$, respectively. Express the covariance matrix $\mathbf{K}_{\hat{\mathbf{x}}} = \mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^T]$ of the real-valued vector $\hat{\mathbf{x}}$ as a function of the covariance matrix $\mathbf{K}_x = \mathbb{E}[\mathbf{x}\mathbf{x}^H]$ and the pseudo-covariance matrix $\mathbf{J}_x = \mathbb{E}[\mathbf{x}\mathbf{x}^T]$ of the complex-valued vector $\mathbf{x}$. This result implies that $\mathbf{K}_x$ and $\mathbf{J}_x$ completely characterize the second-order statistics of the complex random vector $\mathbf{x}$.

2. Assume now that $\mathbf{x}$ is also circularly symmetric, i.e., that for any $\theta \in [0, 2\pi)$ the distribution of $\mathbf{x}e^{j\theta}$ is equal to the distribution of $\mathbf{x}$. Show that the covariance matrix of $\hat{\mathbf{x}}$ is completely characterized by the covariance matrix $\mathbf{K}_x$ only.

3. **Bonus question:** What does the result obtained in part 2 tell you about correlation between the real and the imaginary components of the vector $\mathbf{x}$?

   *Hint:* An excellent tutorial on complex-valued random variables by I. E. Telatar is available on the class website.

Problem 4  Gaussian Random Vectors

Let $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I})$ be an $N$-dimensional independent and identically distributed (i.i.d.) jointly circularly symmetric complex Gaussian random vector. Let $\mathbf{U}$ be an $N \times N$ unitary matrix, i.e., $\mathbf{UU}^H = \mathbf{I}$.

1. Show that $\mathbf{U}\mathbf{w}$ has the same distribution as $\mathbf{w}$.

2. Give a geometric interpretation of this result—what geometric operation does a multiplication with a unitary matrix correspond to?

3. Let $w(t)$ be complex-valued white Gaussian noise with power spectral density $\sigma^2$. Let $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ be a set of complex-valued orthonormal waveforms, and define

   \[
   z_i = \langle w, s_i \rangle = \int_{-\infty}^{\infty} w(t)s_i^*(t) dt.
   \]

   The resulting vector $\mathbf{z} = [z_1 \ z_2 \ \ldots \ z_M]^T$ is the projection of $w(t)$ onto the basis spanned by $\{s_i(t)\}$. Find the joint distribution of $\mathbf{z}$.

   *Hint:* An excellent tutorial on Gaussian noise by R. G. Gallager is available on the class website.
Problem 5  Chernoff Bound

The indicator function $I_v(X)$ for a real-valued random variable $X$ is defined as

$$I_v(X) = \begin{cases} 1, & \text{if } X \geq v \\ 0, & \text{if } X < v. \end{cases}$$

1. Argue that for $s \geq 0$

$$e^{sX} \geq e^{sv} I_v(X).$$

To do so, consider the two cases $X \geq v$ and $X < v$ separately.

2. Show that

$$E[e^{sX}] \geq e^{sv} P[X \geq v],$$

and hence

$$P[X \geq v] \leq \min_{s \geq 0} \left( e^{-sv} E[e^{sX}] \right). \quad (5.1)$$

The inequality (5.1) is called the Chernoff Bound.

3. Now assume that $X$ is Gaussian with zero mean and unit variance. The probability $P[X \geq v]$ satisfies

$$P[X \geq v] = \frac{1}{\sqrt{2\pi}} \int_{v}^{\infty} e^{-\frac{t^2}{2}} dt \quad v \geq 0$$

Use (5.1) to show that $Q(v) \leq e^{-v^2/2}$. This is a standard bound on the Q-function, which we will need frequently in the subsequent lectures to approximate the probability of receiving a message in error.

**Hint:** Write out the expectation $E[e^{sX}]$, and use the fact that a PDF always integrates to 1.

Problem 6  Transformation of Random Variables

The following is a brief review on the transformation of random variables.

- Let $X$ be a real-valued random variable with density function $f_X(x)$, let $g$ a continuous function, $g : \mathbb{R} \to \mathbb{R}$, and let $g^{-1} A = \{ x \in \mathbb{R} : g(x) \in A \}$, for any set $A \subseteq \mathbb{R}$. The cumulative distribution function of the random variable $Y = g(X)$ can be computed as follows:

$$P(Y \leq y) = P(g(X) \leq y) = P(g(X) \in (-\infty, y])$$

$$= P(X \in g^{-1}(-\infty, y]) = \int_{g^{-1}(-\infty, y]} f_X(x) dx.$$
Let $X_1$ and $X_2$ be random variables with joint density function $f_{X_1,X_2}(x_1, x_2)$, and let $T : (x_1, x_2) \to (y_1, y_2)$ be a one-to-one mapping taking some domain $D \subseteq \mathbb{R}^2$ into some range $R \subseteq \mathbb{R}^2$, with $T : y_1 = g(x_1, x_2), \ y_2 = h(x_1, x_2)$, and the inverse transform $T^{-1} : x_1 = \gamma(y_1, y_2), \ x_2 = \psi(y_1, y_2)$. The joint probability density function of $Y_1$ and $Y_2$ can now be computed as

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(\gamma(y_1, y_2), \psi(y_1, y_2)) |J(y_1, y_2)| .$$

$J$ denotes the Jacobian of the transformation, given as

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial \gamma}{\partial y_1} & \frac{\partial \psi}{\partial y_1} \\ \frac{\partial \gamma}{\partial y_2} & \frac{\partial \psi}{\partial y_2} \end{vmatrix} .$$

For a derivation, see any standard textbook on probability. Note that the one-to-one mapping is crucial because it guarantees the invertibility of $T$.

The following problem requires to apply these two results to obtain some probability distributions that we will frequently use.

1. Let $Z$ be a Rayleigh random variable with probability density function

$$f_Z(z) = \frac{z}{\sigma^2} \exp \left( -\frac{z^2}{2\sigma^2} \right), \quad z \geq 0 .$$

Compute the probability density function of $Y = Z^2$.

2. Let $X = X_1 + jX_2$ be a circularly symmetric complex Gaussian random variable, with real part $X_1$ and imaginary part $X_2$ satisfying $X_1, X_2 \sim \mathcal{N}(0, \sigma^2)$. Compute the probability density function of

$$Y_1 = \sqrt{X_1^2 + X_2^2}, \quad Y_2 = \arctan \left( \frac{X_2}{X_1} \right) .$$

Note that $Y_1$ is the absolute value of the random variable $X$ and $Y_2$ is the phase of $X$. 