Fundamentals of Wireless Communication

Homework 3
Handout date: April 15, 2014

Problem 1  Estimation of the delay spread and the mean delay time

In this exercise, we consider the problem of estimating the multipath delay spread $\sigma_\tau$ and the mean delay time $\tau$ by sending a very short input signal as probing signal.

1. Consider the WSSUS channel model, with the standard I-O relation expressed in terms of the delay-Doppler spreading function for a deterministic input signal $s(t)$ according to

$$r(t) = \int_\tau \int_\nu S_H(\tau, \nu)s(t-\tau)e^{j2\pi\nu t}d\tau d\nu.$$

Consider an arbitrarily short input signal as probing signal, you can assume that $|s(t)|^2 \approx \delta(t)$. Explain how the multipath delay spread and the mean delay time can be measured. You may assume that you can take arbitrarily many measurements $r(t)$, i.e., you have arbitrarily many spreading function realizations to average over.

Hint: You may start by computing the expected power of the output, $\mathbb{E}[|r(t)|^2]$, as a function of time, and explain how the multipath delay spread and mean delay time can be obtained from that quantity.

2. Next, we investigate how much of the total energy arrives within the interval $[\tau - \alpha \sigma_\tau, \tau + \alpha \sigma_\tau]$ around the mean delay time. In many practical relevant cases, the power-delay profile (PDP) is very well modeled as exponentially decaying in $\tau$, i.e.,

$$q(\tau) = \begin{cases} P\lambda e^{-\lambda\tau}, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}$$

where $P$ is the path loss and $\lambda$ is the decay parameter. Note that $\lambda = 1/\tau = 1/\sigma_\tau$.

Use this model to compute the fraction of received energy that arrives within a window of duration $2\alpha \sigma_\tau$ seconds around the mean delay time, i.e., the fraction of received energy within the interval $[\tau - \alpha \sigma_\tau, \tau + \alpha \sigma_\tau]$. What can you conclude for the case $\alpha = 1$? [Hint: Remember that for $\tau < 0$, $q(\tau) = 0$].

Problem 2  Doppler Spread

Consider the WSSUS channel model, with zero-mean time-varying impulse response $h(t, \tau)$ and input $s(t) = e^{j2\pi f_0 t}$.

1. Compute the corresponding received signal $r(t)$. 

2. Compute the autocorrelation function of the received signal $E[r(t + \Delta t)r^*(t)]$ as a function of the power Doppler profile of the channel, defined as

$$p(\nu) = \int_{\tau} C_{H}(\tau, \nu) d\tau.$$ 

3. Show that $r(t)$ is WSS and compute the power spectral density $S_r(f)$ of $r(t)$. How is $S_r(f)$ related to the channel power Doppler profile?

**Problem 3  WSS and US Assumption**

Consider the channel shown in Figure 3.1. The transmitted signal is scattered by two static scatterers in the environment with random complex amplitudes $\alpha_1$ and $\alpha_2$ such that $E[\alpha_i] = 0$ and $E[|\alpha_i|^2] = 1$ for $i = 1, 2$.

![Image](image.png)

Figure 3.1: Fading channel with two scatterers.

- Assume that both scattered paths have equal delay but different Doppler shifts because the two paths arrive under different angles at the mobile; hence, we can set the delays of the scattered paths to zero. The resulting channel can be described as a linear frequency-invariant system; the corresponding time-varying channel impulse response can be written as

$$h(t, \tau) = \delta(\tau) \left( \alpha_1 e^{j2\pi\nu_1 t} + \alpha_2 e^{j2\pi\nu_2 t} \right).$$

1. Compute the time correlation function of this channel defined as

$$R_h(t + \Delta t, t; \tau_1, \tau_2) = E[h(t + \Delta t, \tau_1)h^*(t, \tau_2)] .$$

2. Verify that the channel is wide-sense stationary in time (i.e. the function $R_h(t + \Delta t, t; \tau_1, \tau_2)$ does not depend on absolute time $t$, but only on $\Delta t$) if the path gains corresponding to different Doppler shifts are uncorrelated, i.e., if $E[\alpha_1 \alpha_2^*] = 0$. Consequently, a WSS channel in time is white in Doppler.

- Assume now that the mobile is at rest, so that there is no Doppler shift. However, the scattered paths have different delays. Then, the channel can be modeled as LTI with time-varying impulse response

$$h(t, \tau) = \alpha_1 \delta(\tau - \tau_1) + \alpha_2 \delta(\tau - \tau_2).$$
1. Compute the frequency correlation function of this channel defined as
\[ R_{H}(t_1, t_2; f + \Delta f, f) = \mathbb{E}[L_H(t_1, f + \Delta f) L_H^*(t_2, f)] . \]

2. Verify that this channel is WSS in frequency (i.e. the function \( R_{H}(t_1, t_2; f + \Delta f, f) \) does not depend on absolute frequency \( f \), but only on \( \Delta f \)) if scattering is uncorrelated (US), i.e., when the path gains corresponding to different delays are uncorrelated, that is \( \mathbb{E}[\alpha_1 \alpha_2^*] = 0 \). Hence, a WSS channel in frequency is white in delay.

**Problem 4**  Left Eigenvectors and Principle of Biorthogonality

A nonzero vector \( y \in \mathbb{C}^n \) is called a left eigenvector of \( A \in \mathbb{C}^{n \times n} \) corresponding to the eigenvalue \( \lambda \) if
\[ y^H A = \lambda y^H . \]

In this exercise, we refer to the eigenvectors defined by the equation
\[ Ax = \lambda x \]
as right eigenvectors.

1. Show that a left eigenvector \( y \) corresponding to the eigenvalue \( \lambda \) of \( A \in \mathbb{C}^{n \times n} \) is a right eigenvector of \( A^H \) corresponding to \( \lambda^* \), and also that \( y^* \) is a right eigenvector of \( A^T \) corresponding to \( \lambda \). Show, through an example, that, even for \( A \in \mathbb{R}^{n \times n} \), right and left eigenvectors need not be the same.

2. Let \( A \in \mathbb{C}^{n \times n} \), and let \( \lambda \) and \( \mu \) be two eigenvalues of \( A \), with \( \lambda \neq \mu \). Show that any left eigenvector \( y \) of \( A \) corresponding to \( \mu \) is orthogonal to any right eigenvector \( x \) of \( A \) corresponding to \( \lambda \). This result is known as the principle of biorthogonality. [Hint: The proof of this result requires to manipulate \( y^H A x \) in two ways.]

**Problem 5**  Detection in Gaussian Noise

1. Consider the binary detection problem
\[ r = \sqrt{E_s} h x + w , \]
where \( x = +1 \) or \( -1 \) with equal probability, \( h \) is a given \( n \)-dimensional real-valued vector of unit norm \( ||h||^2 = 1 \), \( E_s \) is the received energy per symbol, and \( w \) is i.i.d. real Gaussian noise that satisfies \( w \sim \mathcal{N}(0, (N_0/2)I) \). Extract a scalar sufficient statistic as shown in Appendix D of the lecture notes. Then, derive the maximum a posteriori (MAP) detection rule and compute its error probability.

2. Now consider the slightly different input-output relation
\[ r = \sqrt{E_s} h x + z , \]
where \( h \) is now complex-valued with norm \( ||h||^2 = 1 \), and \( z \) is circularly symmetric complex Gaussian \( z \sim \mathcal{CN}(0, K) \), with \( K \) nonsingular satisfying \( \mathbf{K} = \mathbf{A} \mathbf{A}^H \), for a nonsingular matrix \( \mathbf{A} \). Compute a scalar sufficient statistic and the probability of error for the MAP rule [Hint: Argue that \( z = \mathbf{A} w \), where \( w \sim \mathcal{CN}(0, I) \). Then, exploit the fact that \( \mathbf{A} \) is invertible to reduce the problem to a binary detection problem in complex white Gaussian noise].