MIMO Capacity and Antenna Array Design

Hervé Ndoumbé Mbonjo Mbonjo\textsuperscript{1}, Jan Hansen\textsuperscript{2}, and Volkert Hansen\textsuperscript{1}

\textsuperscript{1}Chair of Electromagnetic Theory, University Wuppertal, Fax: +49-202-439-1045, Email: \{mbonjo.hansen\}@uni-wuppertal.de
\textsuperscript{2}Information Systems Laboratory, Stanford University, Stanford, CA 94305, USA, Fax: 650-723-8473, Email: jchansen@stanford.edu

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Abstract—It is generally accepted that correlation reduces the capacity of the MIMO radio channel. Recently, however, it has been pointed out that several sources of correlation exist. Whereas most often in communications, correlation is interpreted in terms of spatial correlations due to the propagation environment, also the coupling between transmit and receive antennas has impact on the capacity of a given communication channel. Against intuition, it has been claimed that in certain cases antenna coupling has a beneficial effect. In this paper, we derive a general expression for the capacity of the MIMO channel in dependence on both antenna coupling and spatial correlation. We show how the capacity of the MIMO channel depends on the eigenvalues of the antenna coupling matrices and study the interplay between spatial correlation and antenna coupling at high SNR regime.

I. INTRODUCTION

It is well documented that correlation between the links of a MIMO channel has a detrimental effect on the MIMO capacity\cite{1}, \cite{2}. In general, several reasons for correlation exist. Whereas most often, correlation is considered to be caused by the propagation environment, also the coupling between transmit and receive antennas has impact on the capacity of the communication channel. The difference between the two from a communications point of view is that spatial correlation is not a-priori known at the transmitter and must be provided by feedback. Correlation caused by coupling between antenna elements, however, can be computed or measured; in fact, the design of antenna arrays should be optimized to support a maximum possible rate.

Intuitively, one would build arrays with minimum coupling in order keep the rank of the channel matrix as large as possible. But recently it has been claimed that in certain cases antenna coupling has a beneficial effect\cite{3}, \cite{6}, \cite{12}. Related experimental results are given in \cite{4}. But none of these papers provides a detailed theoretical analysis of the coupling which could rigorously explain these observations.

An exception is \cite{5}, which investigates mutual coupling via the S-matrix formulation.

In this paper, we derive the MIMO channel capacity with coupling between antenna arrays. We use the Z-matrix formulation common in network theory and can show how coupling between antennas physically affects the otherwise only spatially correlated channel. We give the channel’s correlation matrix as a composition of both spatial correlation and mutual coupling. An analytical analysis in the high-SNR regime demonstrates that the eigenvalues of the coupling matrices are important parameters for the design of optimal antenna arrays. We identify in the cases analyzed here that the capacity behavior is basically governed by two competing properties of mutual coupling. On the one hand, a decreased distance between antenna elements yields to higher correlation, which in turn lets the rank of the channel drop and decreases capacity. On the other hand, antenna coupling leads in general to either an increase or decrease of irradiated and/or received power. Whereas on a power constrained transmit side an increase is not beneficial, on the receive side any free gain in SNR is advantageous. Due to the complex propagation and coupling properties, the interplay between increased correlation and power variation and the underlying parameters such as array geometry and antenna distance is quite complicate. We hence provide simulation results to illustrate our findings.

This paper is organized as follows. In the next section, we derive the MIMO channel matrix with antenna coupling using network theory fundamentals. In Section 3, we give the correlation matrix of this channel, which is composed of both the antenna coupling and the spatial correlation of the channel; from this representation, we derive analytically the dependence of the capacity on the eigenvalues of the coupling matrices. We give simulation examples in Section 4 and conclude in Section 5.

II. MODELLING OF MUTUAL COUPLING IN MIMO CHANNELS

A. System Model

We assume a transmitter array with $M$ and a receiver array with $N$ antennas in a frequency flat-fading MIMO channel. Denoting the input signal as $x$ and the output of the channel as $y$, we have the relation

\begin{equation}
y = Hx + n
\end{equation}

where $H$ is the $N \times M$ MIMO channel matrix and $n$ an additive white gaussian noise. The capacity $C$ of such a channel is known to be \cite{7}

\begin{equation}
C = \max_{Tr(R_{xx})=M} \log_2 \det \left(I_N + \frac{\rho}{M} H R_{xx} H^H \right)
\end{equation}

where $I_N$ is the identity matrix of size $N$, $\rho$ is the signal-to-noise ratio, and $R_{xx} = E[xx^H]$ with $E[\cdot]$ being the expectation, $(\cdot)^H$ the conjugate transpose of a matrix, and $Tr(\cdot)$ the trace operator; $\det(\cdot)$ denotes the determinant and $\log_2(\cdot)$ the logarithm to base 2.

The channel matrix $H$ has correlated random entries. The correlation matrix is

\begin{equation}
R_H = E[\text{vec}(H)\text{vec}(H)^H]
\end{equation}

where vec(.) is the columns vectorizing operator. The correlation matrix $R_H$ describes all correlations within the channel, i.e., both space and antenna correlation. The antennas are part of the channel.
C. Computation of Mutual Impedance

The mutual impedance between two antennas (denoted as antenna 1 and antenna 2) is derived from the reaction and reciprocity theorems [8] and is given as

\[
Z_{21} = -\frac{1}{I_1I_2} \int_{r_1}^{r_2} E_{21}(r_2) J_2(r_2) \, dr_2
\]  

(5)

where \(J_2(r_2)\) is the current distribution on antenna 2 and \(E_{21}(r_2)\) is the electric field produced by the current distribution \(J_2(r_1)\) on antenna 1 along antenna 2. \(I_1\) and \(I_2\) are the input currents of the antennas. Using fundamental electromagnetic theory (Green’s function concept) [8] \(E_{21}(r_2)\) is related to \(J_1(r_1)\) by

\[
E_{21}(r_2) = \int_{r_1}^{r_2} \frac{G}{r} J_1(r_1) \, dr_1.
\]  

(6)

The integrations in (5) resp. in (6) are performed over the entire regions of antenna 2 resp. antenna 1. In (6) \(\frac{G}{r}\) is the Green function for the electric field. Inserting (6) into (5) we obtain

\[
Z_{21} = -\frac{1}{I_1I_2} \int_{r_1}^{r_2} \int_{r_1}^{r_2} \frac{G}{r} J_1(r_1) J_2(r_2) \, dr_1 \, dr_2
\]  

(7)

for the mutual impedance \(Z_{21}\) between antenna 1 and 2. Equations for the Green function \(\frac{G}{r}\) are available for several kind of spaces (free space [9], layered media [10], half space as a special case of layered media). From this it follows that the mutual impedance between two antennas depends on the knowledge of the current distributions on the antennas. Only in some special cases where the current distribution on the antennas is approximately known (for example thin wire dipoles in free space) it is possible to give an analytically closed expression for the mutual impedance between the antennas. In this paper, (7) is evaluated numerically with the Method of Moments (MOM) [11].

D. MIMO Channel Model Including Antenna Coupling

The goal is now to derive from the matrix formulation of the multi-port model those entries of the correlation matrix \(R_{ij}\) defined in (3) which are due to the mutual coupling of the antenna elements. We use that each port \(i = 1, \ldots, N\) of the receiver array is terminated by a load impedance \(Z_{L,i}\), which relates the currents \(I_i\) and voltages \(u_r\) at the receiver array through Ohm’s law,

\[
u_r = -Z_{L,i} I_i.
\]  

(8)

\(Z_L\) represents the diagonal matrix of the load impedances at the receiver array. Inserting (8) into (4) we solve for the linear equation \(u_r = T u_t\) and obtain

\[
T = (I_N + D Z_L^{-1} - B^T A^{-1} B Z_L^{-1})^{-1} B^T A^{-1}.
\]  

(9)

The voltage transfer matrix \(T\) of the MIMO system is now equivalent to the channel matrix \(H\) with antenna coupling. In order to simplify (9), we note that transmitter and receiver are in the far field and make the following assumptions:

- The magnitude of the entries of the mutual impedance matrices \(A\) and \(D\) is much greater than the magnitude of the elements of the transfer impedance matrices \(B\) and \(B^T:\)

\[
B^T A^{-1} B Z_L^{-1} \ll I_N + D Z_L^{-1}.
\]  

(10)
The matrices A, D, and B^T are independent from each other. This is a trivial assumption, but theoretically, in the derived channel model all antenna ports are coupled.

From these two assumptions we obtain a much simpler form of the MIMO channel matrix,

\[ H_c = Z_L(Z_L + D)^{-1}B^T A^{-1}. \] (11)

We transform (11) further in order to separate spatial correlation and antenna coupling. Assume that the arrays (TX or RX) are each composed of identical antennas and define

\[ C_{RX} = Z_L(Z_L + D)^{-1} = c_{RX,11}I_N + C_{RX,12} \] (12)

and

\[ C_{TX} = A^{-1} = c_{TX,11}I_M + C_{TX,12}. \] (13)

Here, \( c_{RX,11}I_N \) is the self-coupling matrix at the RX-array, which is diagonal. If the antenna elements of the RX-array were different, the diagonal entries of \( c_{RX,11}I_N \) would be different and we have to replace \( c_{RX,11}I_N \) by any diagonal matrix. \( C_{RX,12} \) is the mutual coupling matrix at the RX-array. It incorporates the mutual coupling between two antenna elements \( i \) and \( j \) with \( i \neq j \) at the RX-array. The diagonal entries of \( C_{RX,12} \) include the influence of the mutual coupling on the self-coupling. At the TX-array, coupling is equivalently described by \( c_{TX,11}I_M \) and \( C_{TX,12} \).

Using (12) and (13) we obtain for the channel matrix \( H_c \)

\[ H_c = c_{RX,11}I_NB^TC_{TX,11}I_M + c_{RX,11}I_NB^TC_{TX,12} 
+ C_{RX,12}B^TC_{TX,11}I_M + C_{RX,12}B^TC_{TX,12}. \] (14)

Without coupling, \( C_{RX,12} \) and \( C_{TX,12} \) are zero matrices; this simplifies (14) to yield an expression for the channel matrix without coupling, \( H_{nc} \),

\[ H_{nc} = c_{RX,11}I_NB^TC_{TX,11}I_M. \] (15)

Now, we can rewrite \( H_c \), the channel matrix with coupling, as a function of \( H_{nc} \) and coupling matrices,

\[ H_c = H_{nc} + H_{nc}(c_{TX,11}I_M)^{-1}C_{TX,12} 
+ C_{RX,12}(c_{RX,11}I_N)^{-1}H_{nc} 
+ C_{RX,12}(c_{RX,11}I_N)^{-1}H_{nc}(c_{TX,11}I_M)^{-1}C_{TX,12}. \] (16)

We introduce matrices \( K_{TX} \) and \( K_{RX} \) of the form

\[ K_{TX} = (c_{TX,11}I_M)^{-1}C_{TX,12} \] (17)

\[ K_{RX} = C_{RX,12}(c_{RX,11}I_N)^{-1} \] (18)

and end up with the very compact expression

\[ H_c = (I_N + K_{RX})H_{nc}(I_M + K_{TX}). \] (19)

The matrix \( H_{nc} \) is unitless and equivalent to the channel matrix \( H \) given in (1) for a MIMO channel that does not include antenna coupling. The matrices \( K_{TX} \) and \( K_{RX} \) describe the coupling between different antennas at each array, normalized by the self-coupling.

Eq. (19) can be interpreted as follows. Expanding the products, one obtains a sum of four terms. The first, \( I_NH_{nc}I_M \), indicates that the uncoupled channel appears as a full term in the coupled channel. Even in the presence of other antennas, each link between a transmit and a receive antenna still exists. The second and third terms, \( K_{RX}H_{nc} \) and \( H_{nc}K_{TX} \) indicate that links exist where there is coupling only on one side of the array, but the other side transmits/receives as if no other antennas were around. Finally, in the full coupling term, \( K_{RX}H_{nc}K_{TX} \), all paths are affected by all antennas. A schematic representation of this reading of (19) is given in Fig. 2. It should be stressed out that this intuitive interpretation of eq. (19) results from (10). We note that the derived model is by no means intuitive; several researchers have only postulated MIMO channel models including coupling between antenna arrays without any analytical derivation [3], [12].

III. IMPACT OF THE MUTUAL COUPLING ON THE CAPACITY OF THE MIMO CHANNEL

A. Structure of the Correlation Matrix

The fact that the mutual coupling can be factored out of the MIMO matrix \( H \) allows to rewrite the correlation matrix \( R_H \) of the channel. We use [13] [Lemma 2.2.2] for complex matrices to substitute (19) into (3), and find

\[ R_H = \left((I_M + K_{TX})^T \otimes (I_N + K_{RX})\right) 
R_{H,nc} \left((I_M + K_{TX})^* \otimes (I_N + K_{RX})^H\right). \] (20)

where \((.)^*\) denotes conjugation without transposition and \( \otimes \) is the Kronecker product. The matrix \( R_{H,nc} \) is now the spatial correlation matrix without coupling between antennas, i.e.,

\[ R_{H,nc} = \text{cov}(\text{vec}(H_{nc})\text{vec}(H_{nc})^H). \] (21)

For the special case that [7]

\[ R_{H,nc} = R_t^T \otimes R_r \] (22)

where \( R_t \) and \( R_r \) are the transmit and receive correlation matrices, \( R_{H,nc} \) is given by [7]

\[ R_{H,nc} = R_t^{1/2}R_{H} R_r^{1/2} \] (23)

where \( R_{H} \) is the spatially white \( NxM \) MIMO channel, i.e. the entries of \( R_{H} \) are identically independent distributed (i.i.d). Further, we use [13] [Section 2.2] and obtain for the correlation matrix which includes the coupling between antennas

\[ R_H = \left((I_M + K_{TX})^T \otimes (I_N + K_{RX})^* \otimes (I_N + K_{RX})^H\right). \] (24)

The antenna coupling affects both transmit and receive correlation in the same way.
B. High-SNR Analysis of the MIMO Capacity

In order to obtain a better understanding of the impact of the mutual coupling on the MIMO capacity, we include (23) in (19) and approximate (2) for the case of high SNR and \( M = N \),

\[
C \simeq \log_2 \det \left( \frac{\rho}{M} \mathbf{H}_w \mathbf{H}_w^H \right) + \log_2 \det (\mathbf{R}_t) \\
+ \log_2 \det (\mathbf{R}_r) \\
+ \log_2 \det (\mathbf{I}_M + \mathbf{K}_{TX}) (\mathbf{I}_M + \mathbf{K}_{TX})^H \\
+ \log_2 \det (\mathbf{I}_M + \mathbf{K}_{RX}) (\mathbf{I}_M + \mathbf{K}_{RX})^H \\
= C_{nc} + \Delta C  \tag{25}
\]
as long as the matrices in the brackets have full rank. Here, \( C_{nc} \) is the channel capacity without mutual coupling and \( \Delta C \) the resulting difference. We note that the result is basically illustrative and approximative since the perfect power allocation scheme \( \mathbf{R}_{tx} \) is now assumed to be constant [7]. Furthermore, at very small distances, the spatial correlation between the antennas is very high, so that the rank of the spatial correlation matrices is very small and the SNR needs indeed to be high to justify this analysis.

Since the determinant of unitary matrices is one, we can transform the antenna coupling matrices \( \mathbf{I}_M + \mathbf{K}_{TX}, \mathbf{I}_M + \mathbf{K}_{RX} \) into diagonal matrices and evaluate the determinant,

\[
\det (\mathbf{I}_M + \mathbf{K}_{TX}) (\mathbf{I}_M + \mathbf{K}_{TX})^H = \prod_{i=1}^{M} \lambda_{i, TX} \tag{26}
\]

\[
\det (\mathbf{I}_M + \mathbf{K}_{RX}) (\mathbf{I}_M + \mathbf{K}_{RX})^H = \prod_{i=1}^{M} \lambda_{i, RX} \tag{27}
\]

where the \( \lambda_{i, TX} \) and \( \lambda_{i, RX} \) are the eigenvalues of the matrices \( (\mathbf{I}_M + \mathbf{K}_{TX}) (\mathbf{I}_M + \mathbf{K}_{TX})^H \) and \( (\mathbf{I}_M + \mathbf{K}_{RX}) (\mathbf{I}_M + \mathbf{K}_{RX})^H \). Evaluating the logarithm gives the change in capacity due to antenna coupling at high SNR as

\[
\Delta C = \sum_{i=1}^{M} \log_2 (\lambda_{i, TX}) + \sum_{i=1}^{M} \log_2 (\lambda_{i, RX}) . \tag{28}
\]

Since the coupling matrices \( (\mathbf{I}_M + \mathbf{K}_{TX}) (\mathbf{I}_M + \mathbf{K}_{TX})^H \) and \( (\mathbf{I}_M + \mathbf{K}_{RX}) (\mathbf{I}_M + \mathbf{K}_{RX})^H \) are hermite, their eigenvalues are greater than 0, so that \( \Delta C \) is well-defined. One can recognize that, as in the case of spatial correlation only, the eigenvalues of the coupling matrices play a central role in the behavior of the capacity. If there is no channel knowledge available at the transmitter, constant power allocation is superior. Hence, arrays should be designed such that the eigenvalues of their coupling matrices are all about equal, and as large as possible.

Fig. 3 depicts \( \Delta C \) for a MIMO system using \( M = N = 2 \) \( \lambda / 2 \) dipoles in a side by side configuration of the spacings between the TX and RX antennas. The capacity varies both as a function of the transmit and the receive array antenna element spacing. As a general trend, \( \Delta C \) drops the smaller the distance between the antennas is. We attribute this drop to the loss in rank of the coupling matrices. However, at \( 0.5 \leq d_{TX} / \lambda \leq 0.7 \), and in particular with \( d_{RX} = 0.1 \lambda \) (\( \lambda \) is the wavelength), there is a steep increase visible. This effect is caused by an increase in power consumption that comes along with increased coupling between the antennas; it is similar to moving a single antenna close to a reflecting surface, which also changes, e.g., the power received by this antenna. On the receive side, this increase is in fact beneficial, since it comes without cost; on the transmit side, however, it is equivalent to higher transmit power. This change in power will be investigated in more detail in the next section.

IV. Numerical Results

In order to illustrate our analytical results, we perform Monte Carlo simulations with 10000 channel realizations and compute the ergodic capacity \( E[C] \) at 20 dB SNR of a MIMO system with \( M = N = 2 \) antennas. The TX and RX antennas are half wavelength dipoles at 2 GHz in a side by side configuration separated by a distance \( d_{TX} / \lambda \) resp. \( d_{RX} / \lambda \) where \( \lambda \) is the wavelength.

We first consider the impact of antenna coupling on the ergodic capacity for the case of no spatial correlation, i.e., \( \mathbf{R}_t = \mathbf{R}_r = \mathbf{I}_2 \).

We use three different normalizations of \( \mathbf{H}_w \), and hence three different interpretations of the SNR. In the first one, we have

\[
\| \mathbf{H}_{nc} = \mathbf{H}_w \|_F^2 = NM = 4  \tag{29}
\]

where \( \| \cdot \|_F \) is the Frobenius norm. Here, we do not take into account the instantaneous variation of the transmitted and received power due to antenna coupling. Smaller distances between transmit and receive antennas will increase the received power and be beneficial for the capacity. The second normalization uses

\[
\| \mathbf{H}_w \|_F^2 = NM = 4  \tag{30}
\]

The power variation is cancelled and we investigate only the impact of antenna coupling in the rank of the channel matrix. The third normalization is the most relevant for practical comparisons. We normalize

\[
\| \mathbf{H}_w (\mathbf{I}_M + \mathbf{K}_{TX}) \|_F^2 = NM = 4 ,  \tag{31}
\]
i.e., the transmit power remains constant (constant battery life), but we can benefit from increased receive power.

Fig. 4 depicts the curves obtained by (29), (30) and (31) for the ergodic capacity at \( d_{TX} / \lambda = 0.5 \) as function of the spacing \( d_{RX} / \lambda \). Also, the ergodic capacity for no coupling is given as
reference (constant line, $C_{iid}$). For the first norm, we obtain that the capacity gain due to increase in power outperforms the capacity loss due to the reduced rank of the coupling matrices. However, if the power fluctuations are removed (30), the decrease in rank leads to a significant drop in $E[C]$. The ergodic capacity calculated with (31) corresponds to the curve obtained with (29) shifted by about -0.65 bps. This value corresponds to the loss in capacity due to the normalization of the transmit power. We see that in this case, the capacity is almost always below $C_{iid}$, i.e., the coupling reduces the capacity, except for very low $d_{RX}$. However, in this case $C_{iid}$ will in reality still be higher, since spatial correlation also increases with decreasing element spacing.

In order to assess this interaction between spatial correlation and antenna coupling we compute the ergodic capacity of the MIMO system with antenna coupling and spatial correlation. We use the following simplified correlation model

$$ R_t = \left( \begin{array}{c} 1 \\ \varepsilon_{RX} \end{array} \right), \quad R_r = \left( \begin{array}{c} 1 \\ \varepsilon_{TX} \end{array} \right) $$

(32)

for the spatial correlation at the transmitter and receiver array. In (32) $\varepsilon_{TX}$ and $\varepsilon_{RX}$ are complex numbers that can be obtained from any spatial correlation model. Here, we assume that $\varepsilon_{TX}$ and $\varepsilon_{RX}$ are real numbers. We consider two normalizations which are equivalent to (29) and (31), i.e.,

$$ \| R_t^{1/2} H_0 R_t^{1/2} \|_F^2 = NM = 4, \quad \text{and} \quad \| R_r^{1/2} H_w R_r^{1/2} (M + K_{TX}) \|_F^2 = NM = 4. $$

(33)

The obtained curves are depicted in Fig. 5 for $d_{TX}/\lambda = 0.5$ and $\varepsilon_{TX} = 0$, i.e., no spatial correlation at the receiver, and for $d_{RX}/\lambda = 0.2$ and $d_{RX}/\lambda = 0.6$ with variable $\varepsilon_{RX}$ at an SNR of 20 dB. Fig. 5 shows that for both normalizations, a capacity gain over $C_{iid}$ can only be achieved at virtually no spatial correlation, which is for these very close antenna distances not realistic.

V. CONCLUSIONS

In this paper we investigated a MIMO channel that includes antenna coupling and we derive the correlation matrix of the channel matrix. Furthermore we show mathematically that the impact of the coupling on the capacity can easily be studied with the aid of the eigenvalues of the coupling matrices. From the examples shown here it is identified that the MIMO channel capacity with antenna coupling is influenced by two effects if spatial correlation is not considered: 1) reduced element spacing yields loss in the rank of the channel matrix and hence a decrease of the capacity, 2) reduced element spacing yields an increase in transmitted and/or received power and thus an increase of the capacity. Arrays should be designed in order to find a good tradeoff between these two effects. Under transmit power constraints and if also spatial correlation is taken into account, the capacity of the channel without coupling could not be exceeded.

REFERENCES