OVERSAMPLING IN WAVELET SUBSPACES

Helmut Bölcskei

Institute of Communications and Radio-Frequency Engineering, Vienna University of Technology
Gusshausstrasse 25/389, A-1040 Vienna, Austria
phone: +43 1 58801 3527; fax: +43 1 587 05 83; e-mail: hboelcsk@aurora.nt.tuwien.ac.at

Abstract—Recently, several extensions of classical Shannon sampling theory to wavelet subspaces have been reported. This paper is devoted to uniform and periodic nonuniform oversampling in wavelet subspaces. Specifically, we provide a stability analysis and we introduce a technique for calculating the condition number of wavelet subspace sampling operators. It is shown that oversampling results in improved numerical stability. We consider the reconstruction from noisy samples and we characterize compactly supported scaling functions having compactly supported synthesis functions. Finally, it is shown that in the oversampled case the synthesis functions are not uniquely determined.

1 INTRODUCTION AND OUTLINE

The classical Shannon sampling theorem applies to bandlimited signals, i.e., signals in the Paley-Wiener space of bandlimited functions. More general spaces admitting a sampling theory are reproducing kernel Hilbert spaces (RKHS). For a detailed discussion of RKHS in sampling theory see [1] and the references therein. It has been recognized that wavelet (multiresolution) subspaces are RKHS under mild conditions on the scaling function \( \phi(t) \) [2, 3]. Several extensions of classical Shannon sampling theory to wavelet subspaces have been reported [2, 3, 4]. In [2] Walter considers the reconstruction of a signal \( f(t) \) in \( \mathcal{V}_0 \) from its samples \( f(kT) \). In [3] Janssen discusses the case of uniform noninteger critical sampling, i.e., reconstruction from the samples \( f(t_k + \ell T) \) with an arbitrary parameter \( t_0 \in [0, T] \). Using the Zak transform [5, 6] Janssen calculates the condition number of the sampling operator and shows that it depends critically on the choice of \( t_0 \). In [4] Walter’s work is extended to periodic nonuniform sampling, reconstruction from local averages, and oversampling among others. Based on the theory of filter banks [7, 8] Djokovic and Vaidyanathan furthermore provide a unified treatment of generalized sampling in wavelet subspaces [4].

In this paper, extending results reported in [4], we discuss uniform and periodic nonuniform oversampling in wavelet subspaces using the recently developed theory of oversampled filter banks [9, 10, 11, 12]. Specifically, we provide a stability analysis and we introduce a technique for calculating the condition number of wavelet subspace sampling operators. It is shown that oversampling results in improved numerical stability. We consider the reconstruction from noisy data and we show that in the oversampled case, the synthesis functions are not uniquely determined for given scaling function \( \phi(t) \). Finally, we provide a characterization of compactly supported scaling functions having compactly supported synthesis functions.

The paper is organized as follows. In Section 2, we develop a filter bank model for oversampling in wavelet subspaces.

Section 3 provides conditions for perfect reconstruction and discusses stability and uniqueness issues. In Section 4, we consider the reconstruction from noisy data and we present a method for computing the condition number of wavelet subspace sampling operators. In Section 5, we characterize compactly supported scaling functions having compactly supported synthesis functions.

2 RELATION TO OVERSAMPLED FILTER BANKS

Assumptions and preliminary derivations. In the following we assume that \( \{\phi(t - nT)\}_{n \in \mathbb{Z}} \) with \( \phi(t) \) denoting the scaling function constitutes a Riesz basis [13] for the multiresolution subspace \( \mathcal{V}_0 \). We furthermore assume that \( \phi(t) \) is bounded and that

\[
\sum_{n=-\infty}^{\infty} |\phi(t - nT)| < C_\phi
\]

converges uniformly in \([0, T]\) [3]. This assures that \( \phi(t) \in L^1(\mathbb{R}) \). Since \( \{\phi(t - nT)\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \mathcal{V}_0 \), any \( f(t) \in \mathcal{V}_0 \) can be represented as

\[
f(t) = \sum_{n=-\infty}^{\infty} c_n \phi(t - nT)
\]

with a unique sequence \( c_n \in l^1(\mathbb{Z}) \).

The filter bank model. Let us proceed by establishing a filter bank (FB) model for oversampling in wavelet subspaces. We choose \( N \) points \( t_0, t_1, \ldots, t_{N-1} \) in the interval \([0, MT]\) and sample at the instants \( lMT + t_k \) \((k = 0, 1, \ldots, N - 1, l \in \mathbb{Z})\). The signal \( f(t) \) is said to be critically sampled if \( N = M \) and oversampled if \( N > M \). In the special case of uniform sampling the \( t_k \) \((k = 0, 1, \ldots, N - 1)\) are equally spaced in \([0, MT]\). Noting that

\[
f(lMT + t_k) = \sum_{n=-\infty}^{\infty} c_n h_k[lM - n]
\]

with \( h_k[n] = \phi(t_k + nT) \), we can see that the samples \( f(lMT + t_k) \) with \( l \in \mathbb{Z} \) are obtained by filtering the sequence \( c_n \) with \( h_k[n] \) and decimating the filter output by a factor \( M \). It therefore follows that formally the samples \( f(lMT + t_k) \) are the subband signals obtained by passing the coefficient sequence \( c_n \) through the analysis FB depicted in Fig. 1. Consequently, the coefficients \( c_n \) can be reconstructed (see Fig. 1) from the samples \( f(lMT + t_k) \) by means of a synthesis FB according to

\[\text{Note that (1) guarantees that } h_k[n] \in l^1(\mathbb{Z}).\]
with the synthesis filters $g_k[n]$. Inserting (4) into (2) we obtain

$$f(t) = \sum_{k=0}^{N-1} \sum_{n=-\infty}^{\infty} f(\text{IMT} + t_k) g_k[n - \text{IM}] \phi(t - nT),$$

with the synthesis functions $S_{k,i}(t)$. It can easily be shown that $S_{k,i}(t) = S_{k,i}(t - \text{IMT})$ with

$$S_k(t) = \sum_{n=-\infty}^{\infty} g_k[n] \phi(t - nT).$$

The FB in Fig. 1 is said to be critically sampled if $N = M$ and oversampled if $N > M$. Consequently, critical sampling (oversampling) of the signal $f(t)$ is represented by a critically sampled (oversampled) FB. In this paper we focus on the oversampled case $N > M$.

![Fig. 1. Filter bank model of sampling in wavelet subspaces](image)

**Polyphase decomposition.** In the following we shall need the FB's polyphase representation. The polyphase decomposition of the analysis filters $H_k(z) = \sum_{m=-\infty}^{\infty} h_k[m] z^{-m}$ reads

$$H_k(z) = \sum_{m=-\infty}^{\infty} z^{-m} E_k(n) z^{-m} (k = 0,1,\ldots,N-1),$$

where $E_k(n) = \sum_{m=-\infty}^{\infty} h_k[m N - n] z^{-m}$ (k = 0, 1, $\ldots$, $M - 1$) is the nth polyphase component of the $k$th analysis filter $H_k(z)$. The $N \times M$ analysis polyphase matrix is defined as $[E_k(n)]_{m,n} = E_k(n,m)$. The synthesis filters $G_k(z)$ can be similarly decomposed,

$$G_k(z) = \sum_{m=-\infty}^{\infty} z^{-m} R_k(n) z^{-m} (k = 0,1,\ldots,N-1),$$

with the synthesis polyphase components $R_k(n) = \sum_{m=-\infty}^{\infty} R_k[m M - n] z^{-m}$ (k = 0, 1, $\ldots$, $M - 1$). The $M \times N$ synthesis polyphase matrix is defined as $[R_k(m,n)]_{k,n} = R_k(n,m)$.

**Zak transform.** Using the Zak transform (ZT) [5, 6] of $\phi(t)$ defined as

$$Z_\phi(t,z) = \sum_{n=-\infty}^{\infty} \phi(t + nT) z^{-n},$$

it follows that $H_k(z) = Z_\phi(t_k,z)$.

**Sampling and reconstruction operators.** Defining the sampling operator $T$ : $\chi_0 \rightarrow i^r(\mathbb{Z})$ as

$$T : f \rightarrow f(\text{IMT} + t_k), k = 0,1,\ldots,N-1, t \in \mathbb{Z},$$

and the reconstruction operator $S : i^r(\mathbb{Z}) \rightarrow \chi_0$ as

$$S : f(\text{IMT} + t_k) \rightarrow \sum_{n=-\infty}^{\infty} f(t) \phi(t - nT),$$

it can be shown that the analysis polyphase matrix $E(z)$ provides a matrix representation of the analysis operator $T$ and the synthesis polyphase matrix $R(z)$ provides a matrix representation of the synthesis operator $S$ [12, 10]. The samples $f(\text{IMT} + t_k)$ are in $i^r(\mathbb{Z})$ since

$$\sum_{k=0}^{N-1} \sum_{n=-\infty}^{\infty} |f(\text{IMT} + t_k)|^2 \leq \sum_{k=0}^{N-1} |c_k|^2 \sum_{n=-\infty}^{\infty} \phi(t + t_k - nT)^2 < \infty,$$

where we have used the fact that $c_k \in i^r(\mathbb{Z})$ and (1). Note furthermore that in the case of perfect reconstruction (PR) we have

$$ST = I,$$

where $I$ is the identity operator on $\chi_0$.

**3 PERFECT RECONSTRUCTION, STABILITY, AND UNIQUENESS**

**Perfect reconstruction.** The signal $f(t)$ is perfectly recovered from its samples $f(\text{IMT} + t_k)$ if and only if the underlying FB (see Fig. 1) satisfies the PR property. An oversampled FB is a PR system if and only if [11, 10, 12]

$$R(z)E(z) = I_M,$$

where $I_M$ denotes the $M \times M$ identity matrix. Thus, for given scaling function $\phi(t)$ and hence given analysis polyphase matrix $E(z)$ the solution of (8) determines the synthesis filters $g_k[n]$ and through (5) the synthesis functions $S_k(t)$. In the oversampled case the PR synthesis FB is not unique for given analysis FB [11, 10, 12]. Therefore, the synthesis functions $S_k(t)$ will not be unique for given scaling function $\phi(t)$. In practice this design freedom can be exploited to obtain synthesis functions $S_k(t)$ satisfying certain prescribed properties. The complete parameterization of all synthesis FBs providing PR for a given analysis FB presented in [10, 12] can be used to develop a complete parameterization of all synthesis functions $S_k(t)$ providing PR. One specific solution of (8) is the pseudo-inverse of $E(z)$ given by

$$\tilde{R}(z) = [\tilde{E}(z)E(z)^{-1}]^{-1} \tilde{E}(z).$$

The importance of this solution for the reconstruction from noisy data will be discussed in the following section.

**Stability.** An important problem in sampling theory is stable reconstruction. By stability we mean the following: if two sequences of samples satisfy $f_1(\text{IMT} + t_k)$ and $f_2(\text{IMT} + t_k)$ are close in some sense (e.g. $l^2$-norm) then the corresponding reconstructed signals $f_1(t)$ and $f_2(t)$ should be close as well. Defining $f(t) = \sum_{n=-\infty}^{\infty} c_{1,n} \phi(t - nT)$ and $f_2(t) = \sum_{n=-\infty}^{\infty} c_{2,n} \phi(t - nT)$, it is easily shown that

$$\int_{-\infty}^{\infty} |f_1(t) - f_2(t)|^2 dt \leq \|\phi\|^2 \|f\|^2,$$

where $\|\phi\|^2 = \sum_{n=-\infty}^{\infty} |c_{1,n} - c_{2,n}|^2$. Thus, $f_1(t)$ and $f_2(t)$ will be close in $L^2$-sense if the corresponding sequences $c_{1,n}$ and $c_{2,n}$ are close in $l^2$-sense. In practice, stability guarantees that small perturbations in the samples (caused e.g. by quantization or some other modification) result in small perturbations of the output signal. The reconstruction of $c_0$ from the samples $f(\text{IMT} + t_k)$ will be stable if the synthesis operator $S$ is continuous and hence bounded. Now, since

3 Here, $E(z) = E^H (\frac{1}{2})$ with the superscript $H$ denoting conjugate transpose is the paraconjugate of $E(z)$.

4 A linear operator $L : \chi \rightarrow Y$ where $\chi$ and $Y$ are normed.
$S$ is a left-inverse of $T$ (see (7)), $S$ will be continuous if and only if $T$ is bounded below [14], i.e.,

$$
A|f|^2 \leq \sum_{k=0}^{N-1} \sum_{t=-\infty}^{\infty} |f(I(MT + t_k))|^2 \quad \forall f(t) \in \mathcal{V}_0
$$

(9)

with $A > 0$.

Uniqueness. Another important requirement in sampling theory is uniqueness. A sequence of sampling instants is said to be a sequence of uniqueness if $f(IMT + t_k) = g(IMT + t_k)$ implies $f(t) = g(t)$, where $f(t) \in \mathcal{V}_0$ and $g(t) \in \mathcal{V}_0$. Equivalently, uniqueness means that $f(IMT + t_k) = 0$ for $k = 0, 1, \ldots, N - 1$ and $t \in \mathcal{Z}$ implies $f(t) = 0$. Evidently, a sequence of stability is also a sequence of uniqueness. From (9) and (6) it follows that

$$
A|f|^2 \leq \sum_{k=0}^{N-1} \sum_{t=-\infty}^{\infty} |f(IMT + t_k)|^2 = B|f|^2 \quad \forall f(t) \in \mathcal{V}_0
$$

(10)

with the frame bounds $A, B$ and $0 < A < B < \infty$. This means that the underlying FB (see Fig. 1) should implement a frame decomposition in $\ell^2(\mathcal{Z})$ [10, 12, 11].

**Theorem 1.** A sequence of sampling instants is a sequence of stability (and hence a sequence of uniqueness) if and only if the analysis polyphase matrix $E(z)$ has full rank on the unit circle, i.e.,

$$
\text{rank}(E(z)) = M \quad \text{for} \quad 0 < \theta < 1.
$$

**Proof:** The proof follows from the fact that an oversampled FB with $h_k[n] \in \ell^1(\mathcal{Z})$ implements a frame decomposition in $\ell^2(\mathcal{Z})$ if and only if its analysis polyphase matrix $E(z)$ has full rank on the unit circle [10, 12].

# 4 RECONSTRUCTION FROM NOISY DATA AND CONDITION NUMBER

In practice, the robustness of the reconstruction (i.e. the sensitivity to small perturbations in the samples) depends critically on the condition number of the sampling operator. In the following we shall investigate the sensitivity to (quantization) noise $q[n]$ added to the samples $f(IMT + t_k)$ ($k = 0, 1, \ldots, N - 1$). In the noisy case, the input to the synthesis FB is given by

$$
f'(IMT + t_k) = f(IMT + t_k) + q[n]
$$

where the $N$-dimensional vector noise process $q[n] = [q_0[n], q_1[n], \ldots, q_{N-1}[n]]^T$ is assumed to be wide-sense stationary (WSS), zero-mean, uncorrelated and white with identical variances $\sigma_q^2 = E[|q[k]|^2]$ ($k = 0, 1, \ldots, N - 1$). We furthermore assume that reconstruction is performed using $\tilde{R}(z)$. This particular synthesis FB is desirable since it minimizes the reconstruction error variance $\sigma_e^2$ among the class of all FR FBs [12, 15] in the case of white and uncorrelated noise. Defining

$$
A = \text{ess inf} \{0, 1\}, n=0,1, \ldots, M-1 - \lambda_n(\theta),
$$

$$
B = \text{ess sup} \{0, 1\}, n=0,1, \ldots, M-1 - \lambda_n(\theta),
$$

where $\lambda_n(\theta)$ ($n = 0, 1, \ldots, M - 1, 0 < \theta < 1$) denotes the eigenvalues of the matrix $E^H(E(z)^2)E(z)$, we obtain

$$
\frac{1}{B} \leq \frac{\sigma_q^2}{\sigma_e^2} \leq \frac{1}{A}.
$$

(12)

This result says that the reconstruction error variance $\sigma_e^2$ is bounded in terms of the frame bounds $A$ and $B$ and the subband noise variance $\sigma_q^2$. For a fixed $B$ (i.e. fixed lower bound in (12)) it is desirable to have $A$ as large as possible, so that the upper bound is minimized. Since $A \leq B$, we conclude that $A = B$ or equivalently $B/A = 1$ yields optimum behavior from a noise enhancement point of view. We therefore define the condition number

$$
\rho = \frac{B}{A}.
$$

In practice it is desirable to have $\rho \approx 1$. Strictly speaking, the calculation of the condition number requires an eigenanalysis of the matrix $E^H(E(z)^2)E(z)$ for $0 < \theta < 1$. In practice, a pragmatic approach is to perform an eigenanalysis of $E^H(E(z)^2)E(z)$ for $I = 0, 1, \ldots, L - 1$.

In the special case of nonuniform critical sampling, i.e., $N = M = 1$, our FB model reduces to a nondecimated single channel model with analysis filter $H_0(z) = ZQ_t(0, z)$. In this case the condition number is given by

$$
\rho = \frac{\max |ZQ_t(0, e^{j2\pi\rho})|^2}{\min |ZQ_t(0, e^{j2\pi\rho})|^2},
$$

which is the result first reported by Janssen in [3]. Using the fact that $ZT^6$ have zeros in the unit square it follows that there is at least one $t_0 \in [0, T]$ for which $\rho = \infty$ [3]. In fact, it has been shown in [3] that in general the condition number of the single channel scheme depends critically on the choice of $t_0$. Similarly, in the multi-channel schemes discussed in this paper, the condition number depends on $M$ and $t_k$. In fact, for a given scaling function $\phi(t)$ different choices of $M$ and $t_k$ can result in quite different condition numbers. In general oversampling helps to improve the condition number. We shall next investigate the case of uniform and periodic nonuniform oversampling with $M = 1$ and consequently $N > 1$ in more detail. For $M = 1$ the FB in Fig. 1 reduces to a nondecimated system with $H_0(z) = ZQ_t(z, z)$ and hence

$$
E^H(E(z)^2)E(z) = \sum_{k=-\infty}^{\infty} |ZQ_t(z, e^{j2\pi\rho})|^2.
$$

(13)

The condition number is thus given by

$$
\rho = \frac{\max |\sum_{k=-\infty}^{N-1} |ZQ_t(z, e^{j2\pi\rho})|^2|}{\min |\sum_{k=-\infty}^{N-1} |ZQ_t(z, e^{j2\pi\rho})|^2|}.
$$

From (13) it follows that the condition number will be poor if the $ZQ_t(z, e^{j2\pi\rho})$ have near-common zeros. Observe from (13) that increasing the oversampling factor $N$ will in general lead to less variation in $\sum_{k=-\infty}^{N-1} |ZQ_t(z, e^{j2\pi\rho})|^2$ and hence to an improved condition number.

We shall next provide two examples demonstrating how the condition number depends on the sampling instants (Example 1) and how it improves for increasing oversampling factor (Example 2).

**Example 1.** Let the scaling function $\phi(t)$ be the linear spline

$$
\phi(t) = \left\{ \begin{array}{ll}
\frac{1}{T} t, & 0 \leq t < T \\
1 - \frac{1}{T} (t - T), & T \leq t < 2T \\
0, & \text{otherwise} \end{array} \right.
$$

Choosing $M = 1$ and $N = 2$ with $t_0 = \frac{T}{2}$ and $t_1 = \frac{3T}{2}$, we obtain $H_0(z) = \frac{1}{2} + \frac{1}{2} e^{-j2\pi z}$, $H_1(z) = \frac{1}{2} - \frac{1}{2} e^{-j2\pi z}$, and

$$
E^H(E(z)^2)E(z) = |H_0(z)|^2 + |H_1(z)|^2 = |\cos(\pi z/M)|
$$

which yields $\rho = 6.4$. For $t_0 = \frac{T}{2}$ and $t_1 = \frac{3T}{2}$,

$\rho$ is unbounded if $M = 1$ and $N = 2$ with $t_0 = \frac{T}{2}$ and $t_1 = \frac{3T}{2}$.

$6^\text{Under the assumption (1) $ZQ_t(z, e^{j2\pi\rho})$ is continuous under the weak condition that $\phi(t)$ is continuous [3].}$
we have \( H_0(z) = \frac{1}{2} + \frac{z}{2}z^{-1}, H_1(z) = \frac{3}{4} + \frac{1}{4}z^{-1}, \) and \( \rho = 3.2. \)

**Example 2.** In the second example we consider the quadratic spline

\[
\phi(t) = \begin{cases} 
\frac{1}{2} \left( \frac{t}{T} \right)^2, & 0 \leq t < T \\
\frac{3}{2} - \left( \frac{t}{T} \right)^2, & T \leq t < 2T \\
\frac{1}{2} \left( \frac{t}{T} - 3 \right)^2, & 2T \leq t < 3T \\
0, & \text{otherwise}.
\end{cases}
\]

We choose \( N = M = 2 \) (critical sampling) and \( t_0 = \frac{T}{3}, t_1 = \frac{2T}{3}. \) The corresponding condition number is \( \rho = 79.15. \) For the same scaling function and \( N = 2, M = 1 \) (i.e., oversampling by a factor 2) with \( t_0 \) and \( t_1 \) as before we obtain \( \rho = 5.55, \) which is much better than the condition number obtained in the case of critical sampling.

5 COMPACTLY SUPPORTED SYNTHESIS FUNCTIONS

It has first been noted in [4] that unlike the schemes proposed in [2] and [3] oversampling and periodic sampling allow for compactly supported \( \phi(t) \) and compactly supported synthesis functions \( S_k(t). \) Since the pseudo-inverse \( \tilde{R}(z) \) is desirable for reconstruction from noisy data (see Sec. 4), we shall next provide a condition on a compactly supported \( \phi(t) \) to have compactly supported synthesis functions \( S_k(t) = \sum_{n=0}^{\infty} g_n[n] \phi(t - nT) \) with \( g_n[n] \) denoting the synthesis filters corresponding to \( \tilde{R}(z). \) For a compactly supported \( \phi(t), \) the analysis polyphase matrix \( E(z) \) is a polynomial matrix. From (5) it follows that the synthesis functions \( S_k(t) \) will be compactly supported if the synthesis filters \( g_n[n] \) are FIR, in which turn requires that the synthesis polyphase matrix is polynomial.

**Theorem 2.** For a compactly supported scaling function \( \phi(t) \) with corresponding analysis polyphase matrix \( E(z), \) the synthesis functions \( S_k(t) \) corresponding to \( \tilde{R}(z) \) will be compactly supported if and only if the matrix \( E(z)E(z) \) is unimodular, i.e., \( \det[E(z)E(z)] = c \leq 0. \)

**Proof:** In a FB context it has first been shown in [11] that an FIR analysis FB has an FIR pseudo-inverse if and only if \( E(z)E(z) \) is unimodular. Here, we shall give a proof which is different from that provided in [11] and somewhat shorter. From \( \tilde{R}(z) = [E(z)E(z)]^{-1} E(z) \) it is obvious that unimodularity of the matrix \( E(z)E(z) \) is sufficient for the synthesis FB to be FIR if the analysis FB is FIR. We shall next show the necessity. Assume that both \( E(z) \) and \( \tilde{R}(z) \) are polynomial matrices. Then \( E(z)E(z) \) and \( \tilde{R}(z) \tilde{R}(z) \) are polynomial. Since \( \tilde{R}(z) \tilde{R}(z) \) and \( \tilde{R}(z) \tilde{R}(z) \) are polynomial. Since \( \tilde{R}(z) \tilde{R}(z) \) and \( \tilde{R}(z) \tilde{R}(z) \) have to be polynomial. Therefore, the determinant of \( E(z)E(z) \) has to satisfy \( c(z) = \det[E(z)E(z)] = cz^{-K} \) with \( c \in \mathbb{C} \) and \( K \in \mathbb{N}. \)

From Theorem 2 it follows that a compactly supported \( \phi(t) \) is rather unlikely to have compactly supported synthesis functions corresponding to the pseudo-inverse \( \tilde{R}(z). \) In the following theorem we characterize compactly supported scaling functions having compactly supported synthesis functions.

**Theorem 3.** A compactly supported \( \phi(t) \) with corresponding analysis polyphase matrix \( E(z) \) has compactly supported synthesis functions \( S_k(t) \) if and only if rank \( E(z) = M \) for all \( z. \)

**Proof:** The proof follows from Lemma 6.3.5 in [17], which says that a polynomial \( N \times M \) matrix \( E(z) \) with \( N > M \) has a polynomial left-inverse if and only if rank \( E(z) = M \) for all \( z. \)

6 CONCLUSION

We discussed uniform and periodic nonuniform oversampling in wavelet subspaces drawing from the recently established theory of oversampled FBs. Specifically, we provided a method for calculating the condition number of the sampling operator and we showed that oversampling in general improves this condition number. We discussed the reconstruction from noisy data and we pointed out the influence of the condition number on the resulting noise enhancement. We furthermore showed that in the oversampled case the synthesis functions are not uniquely determined and we characterized compactly supported scaling functions having compactly supported synthesis functions.

References