Robust TOA based Localization for Wireless Sensor Networks with Anchor Position Uncertainties

Zemene Walle Mekonnen and Armin Wittneben
Communication Technology Laboratory, ETH Zurich, 8092 Zurich, Switzerland
Email: {mekonnen, wittneben}@nari.ee.ethz.ch

***Abstract*—In this paper, we address the problem of sensor network localization based on Time-of-Arrival (TOA) measurements under the presence of anchor location errors. We consider the practically appealing setup where the agents are low-complexity transmit-only nodes whose clocks' are not synchronized with that of the anchors. The maximum likelihood (ML) solution of the localization problem is formulated and shown to be a non-convex optimization problem. The relaxation of the ML solution to a convex optimization problem based on semi-definite programming (SDP) is proposed. The proposed localization method is compared with the Cramér Rao lower bound (CRLB) and an existing related approach. Simulation results show that the proposed localization method outperforms the existing scheme, notably when the anchor location uncertainties are the dominant source of errors.**

I. Introduction

Location information is increasingly becoming an essential part of emerging wireless sensor network applications such as target tracking, healthcare monitoring and energy-efficient routing, to mention a few [1]. One of the commonly studied sensor network localization methods is based on Time-of-Arrival (TOA) estimation, which in general consists of two types of nodes – anchors (whose positions are known *a priori*) and agents (whose positions are unknown).

The standard assumption in the study of TOA based localization methods is that the locations of the anchors are fixed and perfectly known. However, in practice, the positions of the anchors have to be determined from a calibration step, which is prone to errors. Furthermore, in some application scenarios such as underwater communications, the anchor nodes (which are placed on the water surface) may drift [2]. Similarly, the locations of the anchor nodes in human motion tracking systems [3] , which are placed on the torso, are only known up to some uncertainty. If unaccounted, these anchor location errors may lead to a considerable performance degradation.

Assuming that the clocks of the anchors and the agents are time synchronized, [2] presents a localization method that accounts the anchor position uncertainties by treating the anchor positions as unknowns and estimating them jointly with the agent positions. It is shown that the maximum likelihood (ML) solution of the localization problem is a non-convex optimization problem, and its convex relaxation which is based on semi-definite programming (SDP) is discussed. Motivated by the computational simplicity of second order cone programming (SOCP), in [4] a relaxation of the ML solution based on SOCP is presented.

However, the assumption of clock synchronized agents and anchors requires cooperation among all the nodes that exist in the system, a requirement which severely limits the applicability of such systems [5]. Localization methods that account for anchor location errors and consider a more practical system setup, where the clocks of the anchors are synchronized with each other but the clocks of the agents are free running, are presented in [5], [6]. To cope with the unknown transmit time of the agents, the method in [6] uses Time-Difference-of-Arrival (TDOA) measurements. The shortcomings of TDOA based methods are that the subtraction of pairwise TOA measurements leads to correlated noise and noise enhancement by 3 dB. Inspired by these drawbacks, [5] proposes an approach which directly utilizes the TOA measurements with unknown transmit times. Nevertheless, both [5] and [6] assume the anchor location errors to be small. Given this assumption, the unknown anchor locations are taken out of the estimation process by considering an estimator with an objective function in which the ranges between the anchors and the agents are approximated by their first order Taylor-series expansion around the measured anchor positions. It is reported that when the anchor location errors are significant, the proposed approximation incurs a substantial performance loss.

In this paper, as in [5], we consider a system setup where the anchors are time synchronized but the agents are transmit-only nodes with free-running clocks which are not synchronized with each other and with that of the anchors. Instead of performing some approximations to eliminate the unknown anchor locations from the estimation process, we take a new approach which jointly estimates the anchor locations along with the unknown locations and transmit times of the agents. The ML solution of the localization problem, which is a non-convex optimization problem is presented. One of the main contributions of this paper is the relaxation of the ML solution to a SDP problem, which can be efficiently solved by optimization toolboxes that are readily available in the literature. The Cramér Rao lower bound (CRLB) of the agent location estimates in the presence of unknown timing offsets and uncertain anchor locations is derived. Simulation results show that the proposed localization method outperforms the method presented in [5], notably when the anchor location errors are the dominant source of uncertainties.

The rest of this paper is organized as follows. Section II formally states the localization problem. The ML solution of the problem is presented in Section III. The relaxation of the ML solution to a convex optimization problem based on SDP relaxation is discussed in Section IV. Section V discusses the CRLB of the agent location estimates. The proposed localization method is compared to an existing approach as well as the CRLB in Section VI. Conclusions are drawn in Section VII.

Notations: The matrices $\boldsymbol{I}$ and $\boldsymbol{0}$ denote the identity and the all-zero matrix of appropriate size, respectively. The matrix $\boldsymbol{E}^{(i)}_{M\times N}$ denotes a $M\times N$ matrix with all zero columns except the ith column which contains all ones. $\mathrm{Tr}\{\boldsymbol{A}\}$ represent the trace of matrix $\boldsymbol{A}$. The notation $\boldsymbol{A}\succeq\boldsymbol{B}$ denotes that $\boldsymbol{A}-\boldsymbol{B}$ is a positive semidefinite matrix.

II. Formulation of the Localization Problem

We consider a network which consists of N_r anchors located at $\{\boldsymbol{r}_n\}_{n=1}^{N_r}$ and N_t agents located at $\{\boldsymbol{t}_m\}_{m=1}^{N_t}$. The clocks of the anchors are assumed to be synchronized with each other but not with that of the agents. The range[1] measurement between anchor n and agent m can hence be expressed as

$$d_{mn}=\|\boldsymbol{t}_m-\boldsymbol{r}_n\|+b_m+e_{mn},$$

where b_m is the range measurement offset due to the unknown transmit time of agent m, e_{mn} is the range measurement error. We assume that the agents and the anchors are in a line of sight (LOS), and hence, the range measurement error can be (approximately) modeled as $e_{mn}\sim\mathcal{N}(0,\sigma^2_{mn})$, where $\sigma^2_{mn}=c^2/(4\pi^2\beta^2\mathrm{SNR}_{mn})$, β is the effective bandwidth of the transmit signal and SNR_{mn} is the signal to noise ratio of the link [7]. Furthermore, the range measurement errors between different agents and anchors are assumed to be statistically independent.

The anchor locations are known up to some uncertainties. To account for this, we model the measured location of anchor n as

$$\tilde{\boldsymbol{r}}_n=\boldsymbol{r}_n+\boldsymbol{\delta}_n,$$

where $\boldsymbol{\delta}_n\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{\Delta}_n)$ is the location error of the nth anchor. We further assume that the location errors of the anchors are statistically independent of each other and of the range measurement errors.

Let $\boldsymbol{d}=[d_{11},d_{12},\ldots,d_{N_rN_t}]^T$ denote all the measured ranges and $\tilde{\boldsymbol{r}}=[\tilde{\boldsymbol{r}}_1^T,\ldots,\tilde{\boldsymbol{r}}_{N_r}^T]^T$ represents the measured anchor locations. With this formulation the problem that we have to solve is: estimate the location of the agents $\boldsymbol{t}=[\boldsymbol{t}_1^T,\ldots,\boldsymbol{t}_{N_t}^T]^T$ given $\boldsymbol{d}$ and $\tilde{\boldsymbol{r}}$ under the presence of unknown nuisance ranging offsets $\boldsymbol{b}=[b_1,\ldots,b_{N_t}]^T$ and anchor locations $\boldsymbol{r}=[\boldsymbol{r}_1^T,\ldots,\boldsymbol{r}_{N_r}^T]^T$.

III. Maximum Likelihood (ML) Solution

Noting that the range and the anchor location measurements are statistically independent, the joint probability density function (PDF) of the observations conditioned on the unknown parameters can be expressed as

$$\begin{aligned}p(\boldsymbol{d},\tilde{\boldsymbol{r}}|\boldsymbol{t},\boldsymbol{b},\boldsymbol{r})&=p(\boldsymbol{d}|\boldsymbol{t},\boldsymbol{b},\boldsymbol{r})\,p(\tilde{\boldsymbol{r}}|\boldsymbol{r})\\&=\frac{|\boldsymbol{\Sigma}|^{-\frac{1}{2}}}{(2\pi)^{\frac{N_rN_t}{2}}}\exp\left(-\tfrac{1}{2}(\boldsymbol{d}-\boldsymbol{g}-\boldsymbol{\Gamma}\boldsymbol{b})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{d}-\boldsymbol{g}-\boldsymbol{\Gamma}\boldsymbol{b})\right)\\&\quad\times\frac{|\boldsymbol{\Delta}|^{-\frac{1}{2}}}{(2\pi)^{\frac{3N_r}{2}}}\exp\left(-\tfrac{1}{2}(\tilde{\boldsymbol{r}}-\boldsymbol{r})^T\boldsymbol{\Delta}^{-1}(\tilde{\boldsymbol{r}}-\boldsymbol{r})\right),\end{aligned}$$

where $\boldsymbol{\Sigma}$ and $\boldsymbol{\Delta}$ represent the covariance matrices of the range measurements and the anchor location measurements, respectively, the vector $\boldsymbol{g}=[\|\boldsymbol{t}_1-\boldsymbol{r}_1\|,\ldots,\|\boldsymbol{t}_{N_t}-\boldsymbol{r}_{N_r}\|]^T$ denotes the trial ranges and $\boldsymbol{\Gamma}$ is a $N_rN_t\times N_t$ matrix which is defined as $\boldsymbol{\Gamma}=\left[{\boldsymbol{E}^{(1)}_{N_r\times N_t}}^T,\ldots,{\boldsymbol{E}^{(N_t)}_{N_r\times N_t}}^T\right]^T$.

The ML estimate of the unknown parameters hence takes the form

$$\begin{aligned}&\max_{\boldsymbol{t},\boldsymbol{b},\boldsymbol{r}}\,p(\boldsymbol{d},\tilde{\boldsymbol{r}}|\boldsymbol{t},\boldsymbol{b},\boldsymbol{r})\\&\equiv\min_{\boldsymbol{t},\boldsymbol{b},\boldsymbol{r}}(\boldsymbol{d}-\boldsymbol{g}-\boldsymbol{\Gamma}\boldsymbol{b})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{d}-\boldsymbol{g}-\boldsymbol{\Gamma}\boldsymbol{b})\\&\qquad+(\tilde{\boldsymbol{r}}-\boldsymbol{r})^T\boldsymbol{\Delta}^{-1}(\tilde{\boldsymbol{r}}-\boldsymbol{r}).\end{aligned}\tag{1}$$

We see that, as the elements of $\boldsymbol{g}$ are nonlinear non-convex functions, the ML estimator in (1) is a non-convex optimization problem. Since the objective function admits multiple local minima, a brute-force local search would require a very good initialization which is "close enough" to the global minimum. As an alternative the ML estimator can be realized by stochastic optimization methods such as genetic algorithm and simulated annealing, but they are computationally intensive with no guarantee of attaining the global minimum [2].

In the next section, we present the relaxation of the ML estimator into a SDP problem, which can be efficiently solved by optimization toolboxes that are readily available in the literature.

IV. SDP Relaxation of the ML Solution

To facilitate the discussion of the SDP relaxation procedure, we shall first reformulate the ML estimator. Using the relation $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}=\mathrm{Tr}\{\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^T\}$ and performing some algebraic manipulation, the ML estimator in (1) can equivalently be expressed as

$$\begin{aligned}\min_{\boldsymbol{t},\boldsymbol{b},\boldsymbol{r}}&\,\mathrm{Tr}\left\{\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{d}\boldsymbol{d}^T+(\boldsymbol{g}+\boldsymbol{\Gamma}\boldsymbol{b})(\boldsymbol{g}+\boldsymbol{\Gamma}\boldsymbol{b})^T-2\boldsymbol{d}(\boldsymbol{g}+\boldsymbol{\Gamma}\boldsymbol{b})^T\right)\right\}\\&+\mathrm{Tr}\left\{\boldsymbol{\Delta}^{-1}\left(\tilde{\boldsymbol{r}}\tilde{\boldsymbol{r}}^T+\boldsymbol{r}\boldsymbol{r}^T-2\tilde{\boldsymbol{r}}\boldsymbol{r}^T\right)\right\}.\end{aligned}\tag{2}$$

We note that $(\boldsymbol{g}+\boldsymbol{\Gamma}\boldsymbol{b})(\boldsymbol{g}+\boldsymbol{\Gamma}\boldsymbol{b})^T=[\,\boldsymbol{I}\ \boldsymbol{\Gamma}\,]\begin{bmatrix}\boldsymbol{g}\boldsymbol{g}^T&\boldsymbol{g}\boldsymbol{b}^T\\\boldsymbol{b}\boldsymbol{g}^T&\boldsymbol{b}\boldsymbol{b}^T\end{bmatrix}\begin{bmatrix}\boldsymbol{I}\\\boldsymbol{\Gamma}^T\end{bmatrix}$. Defining $\boldsymbol{Q}=[\,\boldsymbol{I}\ \boldsymbol{\Gamma}\,]$ and introducing some slack variables, the estimator in (2) can equivalently be formulated as

$$\begin{aligned}\min_{\boldsymbol{t},\boldsymbol{r},\boldsymbol{b},\boldsymbol{g},\boldsymbol{Z},\boldsymbol{R}}&\,\mathrm{Tr}\left\{\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{d}\boldsymbol{d}^T+\boldsymbol{Q}\boldsymbol{Z}\boldsymbol{Q}^T-2\boldsymbol{d}(\boldsymbol{g}+\boldsymbol{\Gamma}\boldsymbol{b})^T\right)\right\}\\&+\mathrm{Tr}\left\{\boldsymbol{\Delta}^{-1}\left(\tilde{\boldsymbol{r}}\tilde{\boldsymbol{r}}^T+\boldsymbol{R}-2\tilde{\boldsymbol{r}}\boldsymbol{r}^T\right)\right\},\\\text{s.t.}\quad&\boldsymbol{Z}=\begin{bmatrix}\boldsymbol{g}\boldsymbol{g}^T&\boldsymbol{g}\boldsymbol{b}^T\\\boldsymbol{b}\boldsymbol{g}^T&\boldsymbol{b}\boldsymbol{b}^T\end{bmatrix},\\&\boldsymbol{R}=\boldsymbol{r}\boldsymbol{r}^T,\\&\boldsymbol{g}[l]=\|\boldsymbol{t}_m-\boldsymbol{r}_n\|,\\&\text{with } l=(m-1)N_r+n,\forall m,\forall n,\end{aligned}\tag{3}$$

[1]As a range measurement, we consider the TOA measurement multipled by the speed of light $c\approx3\times10^8m/s^2$, which sometimes is referred to as pseudo-range in the literature.

where we have used the notation $\boldsymbol{g}[l]$ to denote the lth element of vector $\boldsymbol{g}$. Noting the relation between $\boldsymbol{Z}$ and $\boldsymbol{g}$, we see that the lth diagonal element of $\boldsymbol{Z}$ is given by

$$\boldsymbol{Z}[l,l] = (\boldsymbol{g}[l])^2 = \boldsymbol{r}_n^T\boldsymbol{r}_n + \boldsymbol{t}_m^T\boldsymbol{t}_m - 2\boldsymbol{t}_m^T\boldsymbol{r}_n.$$

Let us further define $\boldsymbol{X} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{N_r}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_{N_t}]$ and $\boldsymbol{Y} = \boldsymbol{X}^T\boldsymbol{X}$. We see that the lth diagonal element of $\boldsymbol{Z}$ is related to the elements of $\boldsymbol{Y}$ by

$$\boldsymbol{Z}[l,l] = \boldsymbol{Y}[n,n] + \boldsymbol{Y}[N_r+m, N_r+m] - 2\boldsymbol{Y}[n, N_r+m].$$

From the definition of $\boldsymbol{Y}$ and $\boldsymbol{R}$, we further note that

$$\text{Tr}\{\boldsymbol{R}\} = \text{Tr}\{\boldsymbol{Y}[1:N_r, 1:N_r]\},$$

where $\boldsymbol{Y}[1:N_r, 1:N_r]$ represents a submatrix that consist of the first N_r rows and columns of $\boldsymbol{Y}$.

Using the newly defined variables and applying the above relationships, the estimator in (3) can be expressed as

$$\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{b},\boldsymbol{g},\boldsymbol{Z},\boldsymbol{R},\boldsymbol{Y}} \ & \text{Tr}\left\{\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{d}\boldsymbol{d}^T + \boldsymbol{Q}\boldsymbol{Z}\boldsymbol{Q}^T - 2\boldsymbol{d}(\boldsymbol{g}+\boldsymbol{\Gamma}\boldsymbol{b})^T\right)\right\} \\
& + \text{Tr}\left\{\boldsymbol{\Delta}^{-1}\left(\tilde{\boldsymbol{r}}\tilde{\boldsymbol{r}}^T + \boldsymbol{R} - 2\tilde{\boldsymbol{r}}\boldsymbol{r}^T\right)\right\}, \\
\text{s.t.} \ & \boldsymbol{Z} = \begin{bmatrix} \boldsymbol{g}\boldsymbol{g}^T & \boldsymbol{g}\boldsymbol{b}^T \\ \boldsymbol{b}\boldsymbol{g}^T & \boldsymbol{b}\boldsymbol{b}^T \end{bmatrix}, \\
& \boldsymbol{R} = \boldsymbol{r}\boldsymbol{r}^T, \ \boldsymbol{Y} = \boldsymbol{X}^T\boldsymbol{X}, \\
& \text{Tr}\{\boldsymbol{R}\} = \text{Tr}\{\boldsymbol{Y}[1:N_r, 1:N_r]\}, \\
& \boldsymbol{Z}[l,l] = \boldsymbol{Y}[n,n] + \boldsymbol{Y}[N_r+m, N_r+m] \\
& \qquad - 2\boldsymbol{Y}[n, N_r+m], \\
& \text{with } l = (m-1)N_r + n, \forall m, \forall n, \qquad (4)
\end{aligned}$$

where, for the sake of notational simplicity, we have not included the trivial relationship between $\boldsymbol{r}$ and $\boldsymbol{X}$. We note that the first three matrix equality constraints in (4) are non-convex, and hence the optimization problem is non-convex.

We now relax (4) to a convex optimization problem as follows. We apply a semidefinite relaxation and replace the matrix equality constraint on $\boldsymbol{Z}$ by

$$\boldsymbol{Z} \succeq \begin{bmatrix} \boldsymbol{g}\boldsymbol{g}^T & \boldsymbol{g}\boldsymbol{b}^T \\ \boldsymbol{b}\boldsymbol{g}^T & \boldsymbol{b}\boldsymbol{b}^T \end{bmatrix},$$

which is a convex (but non-linear) matrix inequality constraint. These non-linear matrix inequality constraint can equivalently be expressed as a linear matrix equality constraint of the form [8]

$$\begin{bmatrix} \multicolumn{2}{c}{\boldsymbol{Z}} & \begin{matrix}\boldsymbol{g}\\ \boldsymbol{b}\end{matrix} \\ \boldsymbol{g}^T & \boldsymbol{b}^T & 1 \end{bmatrix} \succeq \boldsymbol{0}.$$

Applying a similar procedure to the other non-convex constraints, (4) can be relaxed to a SDP problem of the form

$$\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{b},\boldsymbol{g},\boldsymbol{Z},\boldsymbol{R},\boldsymbol{Y}} \ & \text{Tr}\left\{\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{d}\boldsymbol{d}^T + \boldsymbol{Q}\boldsymbol{Z}\boldsymbol{Q}^T - 2\boldsymbol{d}(\boldsymbol{g}+\boldsymbol{\Gamma}\boldsymbol{b})^T\right)\right\} \\
& + \text{Tr}\left\{\boldsymbol{\Delta}^{-1}\left(\tilde{\boldsymbol{r}}\tilde{\boldsymbol{r}}^T + \boldsymbol{R} - 2\tilde{\boldsymbol{r}}\boldsymbol{r}^T\right)\right\}, \\
\text{s.t.} \ & \begin{bmatrix} \multicolumn{2}{c}{\boldsymbol{Z}} & \begin{matrix}\boldsymbol{g}\\ \boldsymbol{b}\end{matrix} \\ \boldsymbol{g}^T & \boldsymbol{b}^T & 1 \end{bmatrix} \succeq \boldsymbol{0}, \\
& \begin{bmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \boldsymbol{r}^T & 1 \end{bmatrix} \succeq \boldsymbol{0}, \ \begin{bmatrix} \boldsymbol{Y} & \boldsymbol{X}^T \\ \boldsymbol{X} & \boldsymbol{I} \end{bmatrix} \succeq \boldsymbol{0}, \\
& \text{Tr}\{\boldsymbol{R}\} = \text{Tr}\{\boldsymbol{Y}[1:N_r, 1:N_r]\}, \\
& \boldsymbol{Z}[l,l] = \boldsymbol{Y}[n,n] + \boldsymbol{Y}[N_r+m, N_r+m] \\
& \qquad - 2\boldsymbol{Y}[n, N_r+m], \\
& \text{with } l = (m-1)N_r + n, \forall m, \forall n. \qquad (5)
\end{aligned}$$

The estimator in (5) is now a convex problem which can be efficiently solved by standard convex optimization algorithms such as SeDuMi and SDPT3 [9].

To get some intuition about the impact of the SDP relaxation, consider the constraint $\boldsymbol{Y} = \boldsymbol{X}^T\boldsymbol{X}$. Since we are considering 3D localization, this constraint binds $\boldsymbol{Y}$ to have a maximum rank of 3. When the matrix equality constraint is relaxed to $\boldsymbol{Y} \succeq \boldsymbol{X}^T\boldsymbol{X}$, we are in effect allowing $\boldsymbol{Y}$ to take on higher dimensions. In fact the solution of (5) converges to the maximum rank solution [10]. However, $\boldsymbol{X}$ can only have a maximum rank of 3 and hence the values of $\boldsymbol{Y}$ which are "leaked" to the dimensions higher than 3 are not captured in $\boldsymbol{X}$. This effect is more pronounced when the agents are located outside the convex hull of the anchor locations, which is supported by the simulation results discussed in Section VI.

One approach to cope with this problem is to apply regularization methods. For a problem of similar type, in [10], a regularization term that penalizes a higher dimensional $\boldsymbol{Y}$ is added to the objective function of the estimator. Another approach is to refine the estimates from (5) by applying a local gradient search on (1). However, the later approach is feasible only if the underlying hardware can afford to implement a gradient method. A trade-off between the two approaches, which needs further investigation, is not covered in this work. In this paper, we evaluate the tightness of the relaxation in (5) by comparing it with the CRLB.

V. Theoretical Performance Bound

In this section, we present the CRLB of the agent location estimates, which can be used as a benchmark to analyze the performance of the estimator in (5).

The Fisher information matrix (FIM) of the unknown parameters $\boldsymbol{\theta} = \left[\boldsymbol{t}^T, \boldsymbol{b}^T, \boldsymbol{r}^T\right]^T$ is given by

$$\begin{aligned}
\boldsymbol{J} &= \mathbb{E}\left\{\left(\frac{\partial \ln p\left(\boldsymbol{d}, \tilde{\boldsymbol{r}}|\boldsymbol{\theta}\right)}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial \ln p\left(\boldsymbol{d}, \tilde{\boldsymbol{r}}|\boldsymbol{\theta}\right)}{\partial \boldsymbol{\theta}}\right\} \\
&= \begin{bmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 \\ \boldsymbol{J}_2^T & \boldsymbol{J}_3 \end{bmatrix}, \qquad (6)
\end{aligned}$$

with

$$\begin{aligned}
\boldsymbol{J}_1 &= \mathbb{E}\left\{\left(\tfrac{\partial \ln p}{\partial \boldsymbol{t}}\right)^T \tfrac{\partial \ln p}{\partial \boldsymbol{t}}\right\}, \\
\boldsymbol{J}_2 &= \left[\mathbb{E}\left\{\left(\tfrac{\partial \ln p}{\partial \boldsymbol{t}}\right)^T \tfrac{\partial \ln p}{\partial \boldsymbol{b}}\right\} \quad \mathbb{E}\left\{\left(\tfrac{\partial \ln p}{\partial \boldsymbol{t}}\right)^T \tfrac{\partial \ln p}{\partial \boldsymbol{r}}\right\}\right], \text{ and} \\
\boldsymbol{J}_3 &= \begin{bmatrix} \mathbb{E}\left\{\left(\tfrac{\partial \ln p}{\partial \boldsymbol{b}}\right)^T \tfrac{\partial \ln p}{\partial \boldsymbol{b}}\right\} & \mathbb{E}\left\{\left(\tfrac{\partial \ln p}{\partial \boldsymbol{b}}\right)^T \tfrac{\partial \ln p}{\partial \boldsymbol{r}}\right\} \\ \mathbb{E}\left\{\left(\tfrac{\partial \ln p}{\partial \boldsymbol{r}}\right)^T \tfrac{\partial \ln p}{\partial \boldsymbol{b}}\right\} & \mathbb{E}\left\{\left(\tfrac{\partial \ln p}{\partial \boldsymbol{r}}\right)^T \tfrac{\partial \ln p}{\partial \boldsymbol{r}}\right\} \end{bmatrix}.
\end{aligned}$$

Taking the partial derivative of the log-likelihood with respect to $\boldsymbol{t}$ gives us

$$\frac{\partial \ln p\left(\boldsymbol{d}, \tilde{\boldsymbol{r}}|\boldsymbol{\theta}\right)}{\partial \boldsymbol{t}} = \left(\boldsymbol{d} - \boldsymbol{g} - \boldsymbol{\Gamma}\boldsymbol{b}\right)^T \boldsymbol{\Sigma}^{-1} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{t}}.$$

The first block of the FIM is hence given by

$$\boldsymbol{J}_1 = \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{t}}\right)^T \boldsymbol{\Sigma}^{-1} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{t}},$$

where the relation $\mathbb{E}\{(\boldsymbol{d}-\boldsymbol{g}-\boldsymbol{\Gamma b})(\boldsymbol{d}-\boldsymbol{g}-\boldsymbol{\Gamma b})^T\} = \boldsymbol{\Sigma}$ is used. The term $\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{t}}$ can be calculated straightforwardly. The other building blocks of the FIM can also be calculated similarly, which are not discussed here due to space limitations.

Given the FIM $\boldsymbol{J}$, the covariance matrix of any unbiased estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ satisfies the information inequality

$$\mathbb{E}\left\{\left(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)\left(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^T\right\} \succeq \boldsymbol{J}^{-1}. \qquad (7)$$

Noting that the Schur complement of the matrix $\boldsymbol{J}_3$ in (6) is $\boldsymbol{J}_1 - \boldsymbol{J}_2\boldsymbol{J}_3^{-1}\boldsymbol{J}_2^T$, the inverse of the FIM takes the form [11]

$$\boldsymbol{J}^{-1} = \begin{bmatrix} (\boldsymbol{J}_1 - \boldsymbol{J}_2\boldsymbol{J}_3^{-1}\boldsymbol{J}_2^T)^{-1} & \boxtimes \\ \boxtimes & \boxtimes \end{bmatrix},$$

where $\boxtimes$ represents a block matrix whose value we are not interested in. Applying this to (7) and considering only the information inequality that involves the agent locations $\boldsymbol{t}$, we see that the mean squared error (MSE) of any unbiased estimate $\hat{\boldsymbol{t}}$ of $\boldsymbol{t}$ satisfies

$$\mathbb{E}\left\{\left(\hat{\boldsymbol{t}}-\boldsymbol{t}\right)\left(\hat{\boldsymbol{t}}-\boldsymbol{t}\right)^T\right\} \succeq (\boldsymbol{J}_1 - \boldsymbol{J}_2\boldsymbol{J}_3^{-1}\boldsymbol{J}_2^T)^{-1}. \qquad (8)$$

From (8) we note that the term $\boldsymbol{J}_1$ is the FIM of the agent locations if the anchor locations are known perfectly and the clocks of the agents and the anchors are synchronized perfectly (and hence the ranging offsets are zero and known). The term $\boldsymbol{J}_2\boldsymbol{J}_3^{-1}\boldsymbol{J}_2^T$ represents the reduction of information about the agent locations that is incurred due to the unknown ranging offsets and the uncertainties in the anchor locations.

VI. Performance Evaluation

In this section, we evaluate the performance of the estimator in (5) by comparing it with the method reported in [5] and the theoretical performance bound presented in the previous section. Before we proceed with the discussion of the simulation results, we make a remark that, in [5], two SDP based localization methods were presented. Namely, robust 2-step least squares (R2LS) and robust min-max algorithm (RMMA). Since in our system model we assume independent and Gaussian distributed range measurement errors, we chose the R2LS algorithm for comparison, which is reported as the method that shows a better performance for the considered error model. To solve the proposed estimator in (5) and the R2LS algorithm, we utilize the CVX Matlab toolbox from [9], where the solver SDPT3 is implemented.

As a performance metric, we use the root mean squared error (RMSE) of the agent location estimates, where the "mean" refers to averaging over multiple agents, different agent and anchor topologies and different realizations of the ranging and anchor location errors.

We consider a network consisting of $N_r = 8$ anchors and $N_t = 10$ agents. Both the anchor location errors and the range measurement errors are assumed to be i.i.d. zero-mean Gaussian random variables with a standard deviation of σ_{anchor} and σ_{range}, respectively.

Fig. 1 shows RMSE performance of the proposed SDP based localization method and the R2LS algorithm presented in [5] for varying ratio of $\sigma_{\text{range}}/\sigma_{\text{anchor}}$. Also shown in the figure is the corresponding CRLB. We fix $\sigma_{\text{anchor}} = 5\,\text{cm}$ and the σ_{range} is varied from $1\,\text{cm}$ to $10\,\text{cm}$. Both the anchor locations and the agent locations are drawn randomly [2] from a $10\,\text{m} \times 10\,\text{m} \times 10\,\text{m}$ cube, centered at the origin.

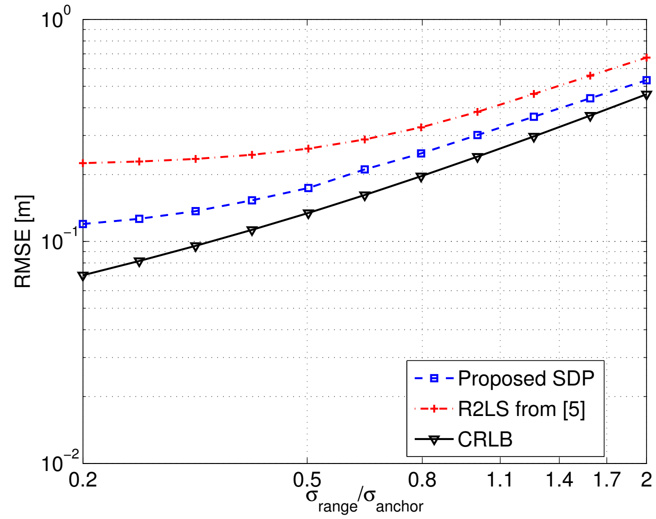

Fig. 1: Comparison of the RMSE performance of the proposed localization method and the R2LS algorithm proposed in [5].

From the figure, we see that the SDP based localization method proposed in this paper outperforms the R2LS algorithm from [5]. The performance improvement is more pronounced when $\sigma_{\text{range}}/\sigma_{\text{anchor}}$ is small (i.e. the anchor location uncertainties are the dominant source of errors) and it diminishes as the ratio $\sigma_{\text{range}}/\sigma_{\text{anchor}}$ increases. This is expected because when the range measurement error variance increases, the first term of the objective function in (5) dominates the objective function value and hence the R2LS algorithm, which uses the first order Taylor series approximation of the objective function around the measured anchor locations, does not incur a big performance loss. Comparing the RMSE performance with the CRLB, we further see that the proposed SDP relaxation is tight even when the ranging error variance is large.

Fig. 2 shows the RMSE performance for the same choice of parameters as in Fig. 1 but now the agents are located outside the convex hull of the anchor locations. Specifically, the anchors are drawn randomly from a $4\,\text{m} \times 4\,\text{m} \times 4\,\text{m}$ cube and the agents are picked randomly from a volume outside this cube but inside a $10\,\text{m} \times 10\,\text{m} \times 10\,\text{m}$ cube. We first note that the RMSE performance is now worse compared to the setup in Fig. 1, which is expected as this node topology is not favorable in terms of localization performance. It is interesting

[2]As the anchor location are chosen randomly, the RMSE values include averaging also over anchor topologies which may not be preferable in terms of localization performance. We consider this to show the robustness of the proposed method.

to see that when the ratio $\sigma_{\text{range}}/\sigma_{\text{anchor}}$ is small the proposed SDP brings a big performance improvement, however the gap diminishes rapidly as σ_{range} increases and even the R2LS algorithm performs better when the ratio $\sigma_{\text{range}}/\sigma_{\text{anchor}}$ is above 1.6.

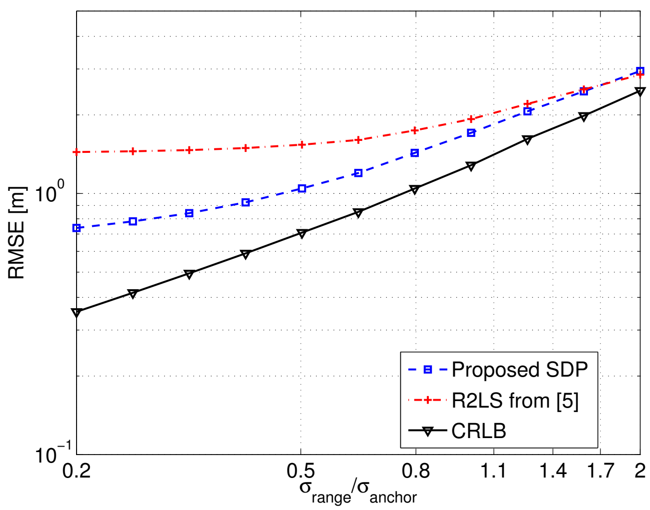

Fig. 2: RMSE performance of the proposed method and the R2LS algorithm proposed in [5] when the agents are located outside the convex hull of the anchors.

Fig. 3 shows the comparison of the two localization methods for a varying number of agents. We set $\sigma_{\text{range}}/\sigma_{\text{anchor}} = 0.2$ (with $\sigma_{\text{anchor}} = 5\,\text{cm}$) and $N_r = 8$. Both the agents and the anchors are drawn randomly from a $10\,\text{m} \times 10\,\text{m} \times 10\,\text{m}$ cube. We see that the proposed scheme, which jointly estimates the anchor locations and the agent locations, benefits as N_t increases. This is because when N_t increases the geometric shape formed by the anchors and the agents becomes more unique which improves the performance of the estimator. On the other hand for the R2LS algorithm, which does not estimate the anchor locations, the estimates of each of the agent's location is uncoupled and hence the algorithm does not benefit from an increase in N_t. It should however be noted that as N_t increases, the estimator in (5) becomes computationally intensive and hence a tradeoff between complexity and accuracy needs to be considered. When the network size is very large, considering a distributed approach, in which the network is divided into smaller sub-networks and the localization problem is solved in a distributed fashion, might be beneficial.

VII. Conclusion and Future Work

We address the problem of sensor network localization based on asynchronous TOA measurements under the presence of anchor location errors. The ML solution of the localization problem, which jointly estimates the unknown anchor locations along with the unknown locations and transmit times of the agents, is relaxed to a convex optimization problem based on SDP. The comparison of the proposed approach with the CRLB shows that the relaxation is tight even when the ranging error variance is large. Our results also show that the proposed scheme brings a performance gain compared to the R2LS algorithm presented in [5], notably when the ratio $\sigma_{\text{range}}/\sigma_{\text{anchor}}$ decreases.

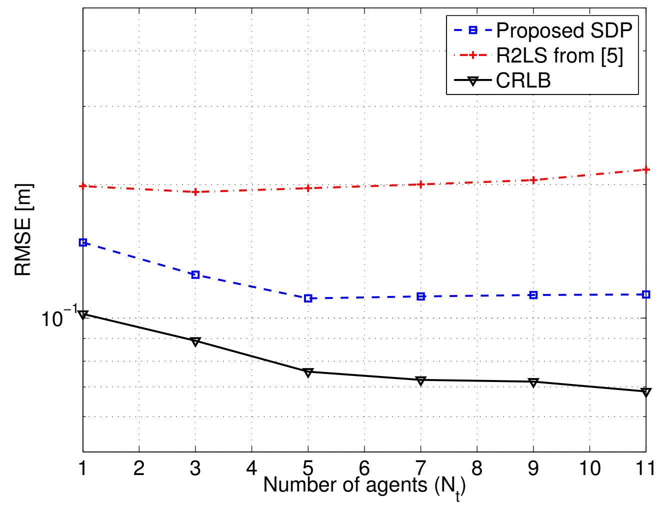

Fig. 3: Impact of the number of agents N_t on the RMSE performance.

Our future work will be directed towards the study of regularization methods that can further improve the proposed estimator in (5). Furthermore, a distributed implementation of the proposed estimator will be addressed.

References

[1] G. Mao, B. Fidan, and B. D. O.Anderson, "Wireless sensor network localization techniques," *Computer Networks*, vol. 51, pp. 2529–2553, July 2007.

[2] K. W. K. Lui, W.-K. Ma, H. So, and F. K. W. Chan, "Semi-definite programming algorithms for sensor network node localization with uncertainties in anchor positions and/or propagation speed," *Signal Processing, IEEE Transactions on*, vol. 57, no. 2, pp. 752–763, February 2009.

[3] Z. Mekonnen, E. Slottke, H. Luecken, C. Steiner, and A. Wittneben, "Constrained maximum likelihood positioning for UWB based human motion tracking," in *Indoor Positioning and Indoor Navigation (IPIN), 2010 International Conference on*, September 2010, pp. 1–10.

[4] G. Shirazi, M. Shenouda, and L. Lampe, "Second order cone programming for sensor network localization with anchor position uncertainty," in *Positioning Navigation and Communication (WPNC), 2011 8th Workshop on*, April 2011, pp. 51–55.

[5] E. Xu, Z. Ding, and S. Dasgupta, "Source localization in wireless sensor networks from signal time-of-arrival measurements," *Signal Processing, IEEE Transactions on*, vol. 59, no. 6, pp. 2887–2897, June 2011.

[6] K. Yang, G. Wang, and Z.-Q. Luo, "Efficient convex relaxation methods for robust target localization by a sensor network using time differences of arrivals," *Signal Processing, IEEE Transactions on*, vol. 57, no. 7, pp. 2775–2784, July 2009.

[7] Y. Qi, "Wireless geolocation in a non-line-of-sight environment," *Ph.D. dissertation, Princeton University*, December 2003.

[8] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in Applied Mathematics. SIAM, 1994, vol. 15.

[9] M. Grant and S. Boyd, "MATLAB software for disciplined convex programming." [Online]. Available: http://cvxr.com

[10] P. Biswas, T.-C. Liang, K.-C. Toh, Y. Ye, and T.-C. Wang, "Semidefinite programming approaches for sensor network localization with noisy distance measurements," *Automation Science and Engineering, IEEE Transactions on*, vol. 3, no. 4, pp. 360–371, October 2006.

[11] F. Zhang, *The Schur Complement and Its Applications*. Springer Science & Business Media, Inc., 2005.