On Capacity Scaling of (Long) MIMO Amplify-and-Forward Multihop Networks

Jörg Wagner and Armin Wittneben
Communication Technology Laboratory, ETH Zurich
{wagner, wittneben}@nari.ee.ethz.ch

***Abstract*— Consider a multiple-input-multiple-output multihop network consisting of $n_\mathcal{S}$ noncooperating source antennas, $n_\mathcal{D}$ fully cooperating destination antennas, and L clusters of $n_\mathcal{R}$ noncooperating relay antennas each. The source signals reach the destination antennas via $L+1$ hops by traversing all relay clusters. Relay antennas scale their received signals by a common constant before retransmission (*amplify-and-forward*). We perform an asymptotic capacity analysis assuming the elements of the (per hop) channel matrices to satisfy the conditions for the Marčenko-Pastur law. Our result: In the limit $L \to \infty$, sum-capacity scales linearly with $\min\{n_\mathcal{S}, n_\mathcal{D}\}$, if and only if L scales no faster than linearly with the ratio $n_\mathcal{R}/\min\{n_\mathcal{S}, n_\mathcal{D}\}$.**

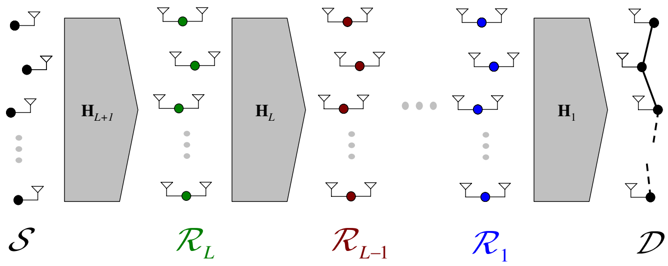

Fig. 1. Network under consideration.

I. Introduction

We consider wireless multiple-input-multiple-output (MIMO) communication between $n_\mathcal{S}$ noncooperating source antennas and $n_\mathcal{D}$ fully cooperating destination antennas. Let us suppose the source antennas are either far apart or shadowed from the destination antennas. The installation of intermediate nodes that relay the source signals to the destination (multihop) is well known for being an efficient means for improving the energy-efficiency of the communication system in this case. In the resulting network, the signals traverse L clusters containing $n_\mathcal{R}$ relay antennas each, before they reach the destination antennas. The relay antennas forward the signals by performing an amplify-and-forward operation. Generally, signals transmitted by the source antennas might not only be received by the immediately succeeding cluster of relay antennas, but possibly also by clusters that are further away. Such receptions could well be exploited for boosting the throughput, but we assume them to be strongly attenuated and ignore them in this paper. A sketch of the network is depicted in Fig. 1. We assume the elements of all single-hop channel matrices to be i.i.d. with zero-mean and unit-variance and that channel state information is only available in the destination cluster.

Previous publications have shown that in such a network linear sum-capacity scaling in $\min\{n_\mathcal{S}, n_\mathcal{D}\}$ is not feasible in the limit of an infinitely long network ($L \to \infty$), when $n_\mathcal{R} \in \Theta(\min\{n_\mathcal{S}, n_\mathcal{D}\})$[1], and the signal-to-noise ratio (SNR) at the destination antennas is fixed [1], [2]. The underlying phenomenon is an unfavorable distortion of the eigenvalue spectrum of the effective channel matrix [1]: almost all eigenvalues tend to zero as $L \to \infty$.

[1] $g(n) \in \mathcal{O}(f(n))$ if $\exists M, n_0 > 0 : M|f(n)| > |g(n)|, \forall n > n_0$,
$g(n) \in \Omega(f(n))$ if $\exists M, n_0 > 0 : M|f(n)| < |g(n)|, \forall n > n_0$,
$g(n) \in \Theta(f(n))$ if $g(n) \in \mathcal{O}(f(n))$ and $g(n) \in \Omega(f(n))$.

By not only taking L, but also the ratio $n_\mathcal{R}/\min\{n_\mathcal{S}, n_\mathcal{D}\}$ to infinity, this paper finds a condition on the relative growth of these quantities which ensures linear sum-capacity scaling for a fixed destination SNR for arbitrarily long networks. This condition is necessary and sufficient, and constitutes a fundamental criterion for the design of amplify-and-forward based multihop systems.

Our work is motivated by the recent studies [3], [4] on the same network for fixed number of hops. In these references it is shown that sum-capacity scales linearly with $\min\{n_\mathcal{S}, n_\mathcal{D}\}$ for any *fixed* L, *fixed* SNR at the destination antennas and $n_\mathcal{R} \in \Omega(\min\{n_\mathcal{S}, n_\mathcal{D}\})$. In particular, up to a pre-log factor, the point-to-point MIMO channel capacity is recovered as $n_\mathcal{R}/\min\{n_\mathcal{S}, n_\mathcal{D}\} \to \infty$. Encouraged by the fact that increasing the ratio $n_\mathcal{R}/\min\{n_\mathcal{S}, n_\mathcal{D}\}$ results in more favorable eigenvalue spectra of the effective channel matrix, we show in this paper that the "eigenvalue loss" due to an increasing number of hops can be compensated by increasing this ratio at least linearly in L.

We remark that only recently it has been shown that vector-quantization based relaying strategies achieve linear sum-capacity scaling with $\min\{n_\mathcal{S}, n_\mathcal{D}\}$ for fixed SNR *and* fixed ratio $n_\mathcal{R}/\min\{n_\mathcal{S}, n_\mathcal{D}\}$ [5]. However, such strategies come along with a significant complexity overhead.

II. I-O Relation & Capacity

Assuming that all relays scale their receive signal by the scalar $\sqrt{\alpha/n_\mathcal{R}}$, $\alpha > 0$, before retransmission, the I-O relation from source to destination cluster is given by [2]

$$\mathbf{y} = \frac{\alpha^{L/2}}{n_\mathcal{R}^{L/2}}\mathbf{H}_1 \cdots \mathbf{H}_{L+1}\mathbf{s} + \mathbf{n}_\mathcal{D} + \sum_{l=1}^{L} \frac{\alpha^{l/2}}{n_\mathcal{R}^{l/2}}\mathbf{H}_1 \cdots \mathbf{H}_l \mathbf{n}_{\mathcal{R}_l},$$

where $\mathbf{H}_l$ denotes the channel matrix relating the transmit vector of relay cluster l (or the source cluster if $l = L+1$)

to the receive vector of relay cluster $l-1$ (or the destination cluster if $l=1$) through the standard point-to-point MIMO I-O relation (frequency-flat fading). The vectors $\mathbf{n}_{\mathcal{R}_l}$ and $\mathbf{n}_{\mathcal{D}}$ refer to complex additive white Gaussian noise of unit variance at the relay antennas in cluster l and in the destination cluster, respectively. The source transmit vector is denoted by $\mathbf{s}$ and due to the lack of channel state information in the source cluster chosen optimally to be a circularly symmetric complex Gaussian random vector with covariance matrix $P/n_{\mathcal{S}} \cdot \mathbf{I}_{n_{\mathcal{S}}}$. We fix $\alpha = P/(P+1)$ in order to allocate the same average transmit power $P/n_{\mathcal{R}}$ to each relay antenna and the same average transmit power P to each relay cluster. The receive signal and noise covariance matrices are given by

$$\mathbf{R}_s = \frac{P\alpha^L}{n_{\mathcal{S}} n_{\mathcal{R}}^L} \mathbf{H}_1 \cdots \mathbf{H}_{L+1} \mathbf{H}_{L+1}^H \cdots \mathbf{H}_1^H,$$

$$\mathbf{R}_n = \mathbf{I}_{n_{\mathcal{D}}} + \sum_{l=1}^{L} \frac{\alpha^l}{n_{\mathcal{R}}^l} \mathbf{H}_1 \cdots \mathbf{H}_l \mathbf{H}_l^H \cdots \mathbf{H}_1^H.$$

We define the empirical eigenvalue distribution (EED) of some Hermitian matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ using the indicator function $1\{\cdot\}$ as $F_{\mathbf{A}}(x) \triangleq \frac{1}{n}\sum_{i=1}^n 1\{\lambda_i\{\mathbf{A}\} < x\}$. Also, we define $\beta_{\mathcal{R}} \triangleq \frac{n_{\mathcal{R}}}{n_{\mathcal{D}}}$ and $\beta_{\mathcal{S}} \triangleq \frac{n_{\mathcal{S}}}{n_{\mathcal{D}}}$. With this notation, the ergodic sum-capacity of the network in nats per channel use and destination antenna is given by

$$C = \frac{1}{L_0} \cdot \mathsf{E}\left[\int_0^\infty \log(1+x) \cdot dF_{\mathbf{R}_s\mathbf{R}_n^{-1}}^{(n_{\mathcal{D}},\beta_{\mathcal{R}},L,\beta_{\mathcal{S}})}(x)\right], \quad (1)$$

where expectation is taken w.r.t. the $\mathbf{H}_l$, $l \in \{1,\ldots,L+1\}$. The superscripts of the EED indicate the parameters it depends on. Whenever we take a parameter from this list to infinity, we will drop the respective superscript in the following. L_0 denotes the number of time slots a cluster is silent after transmission to avoid interference for succeeding clusters.

III. Main Result

We seek for a condition on the relative growth of L with $\beta_{\mathcal{R}}$ that guarantees linear sum-capacity scaling with $\min\{n_{\mathcal{S}}, n_{\mathcal{D}}\}$. Therefore, we *first* need to take $n_{\mathcal{D}}$ (thus also $n_{\mathcal{S}} = \beta_{\mathcal{S}} n_{\mathcal{D}}$ and $n_{\mathcal{R}} = \beta_{\mathcal{R}} n_{\mathcal{D}}$) to infinity for $\beta_{\mathcal{S}}$, $\beta_{\mathcal{R}}$ and L fixed: in doing so, the EED $F_{\mathbf{R}_s\mathbf{R}_n^{-1}}^{(n_{\mathcal{D}},\beta_{\mathcal{R}},L,\beta_{\mathcal{S}})}(\cdot)$, almost surely converges uniformly to the asymptotic EED $F_{\mathbf{R}_s\mathbf{R}_n^{-1}}^{(\beta_{\mathcal{R}},L,\beta_{\mathcal{S}})}(\cdot)$ [3]. Then, we take $\beta_{\mathcal{R}}$ and L (as a function of $\beta_{\mathcal{R}}$) to infinity. Note that these limits do not commute necessarily. Whenever we refer to convergence of an EED to an asymptotic EED in the following, we mean convergence in the almost sure sense.

Theorem. *Let $\mathbf{H}_1 \in \mathbb{C}^{n_{\mathcal{D}}\times n_{\mathcal{R}}}$, $\mathbf{H}_2,\ldots,\mathbf{H}_L \in \mathbb{C}^{n_{\mathcal{R}}\times n_{\mathcal{R}}}$ and $\mathbf{H}_{L+1} \in \mathbb{C}^{n_{\mathcal{R}}\times n_{\mathcal{S}}}$ be random matrices with i.i.d. elements of zero-mean, unit-variance and bounded fourth moment. Let* snr *(SNR at destination antennas) be a positive constant. Fix*

$$P = \left(\sqrt[L+1]{\frac{\mathrm{snr}+1}{\mathrm{snr}}} - 1\right)^{-1} \quad \text{and} \quad \alpha = \frac{P}{P+1}.$$

Then, as $\beta_{\mathcal{R}} \to \infty$, $F_{\mathbf{R}_s\mathbf{R}_n^{-1}}^{(\beta_{\mathcal{R}},L,\beta_{\mathcal{S}})}(\cdot)$ converges pointwise to

$$F_{\mathbf{R}_s\mathbf{R}_n^{-1}}^{(\beta_{\mathcal{S}})}(x) = \begin{cases} F_{\mathrm{MP}}^{(\beta_{\mathcal{S}})}\left(\frac{x}{\mathrm{snr}}\right), & \text{if } L \in \mathcal{O}\left(\beta_{\mathcal{R}}^{1-\varepsilon}\right), \varepsilon > 0, \\ \sigma(x) & \text{, if } L \in \Omega(\beta_{\mathcal{R}}^{1+\varepsilon}), \varepsilon > 0. \end{cases},$$

where $F_{\mathrm{MP}}^{(\beta_{\mathcal{S}})}(\cdot)$ and $\sigma(\cdot)$ denote the Marčenko-Pastur distribution with parameter $\beta_{\mathcal{S}}$ and the unit step, respectively. Moreover, if $L \in \Theta(\beta_{\mathcal{R}})$, $F_{\mathbf{R}_s\mathbf{R}_n^{-1}}^{(\beta_{\mathcal{R}},L,\beta_{\mathcal{R}})}(\cdot)$ converges pointwise to an asymptotic EED $F_{\mathbf{R}_s\mathbf{R}_n^{-1}}^{(\beta_{\mathcal{S}})}(x) \neq \sigma(x)$.

Considering the capacity expression (1), the theorem allows to conclude the following:

- Sum-capacity scales linearly in $\min\{n_{\mathcal{S}}, n_{\mathcal{D}}\}$ as long as L scales no faster than linearly with $\beta_{\mathcal{R}}$.
- Sum-capacity per source (or destination) antenna tends to zero in the limit $n_{\mathcal{D}} \to \infty$ for faster than linear scaling of L with $\beta_{\mathcal{R}}$.
- The point-to-point capacity is approached (up to the pre-log factor) for slower than linear growth of L with $\beta_{\mathcal{R}}$.

The last point follows, since the Marčenko-Pastur distribution [6] corresponds to the asymptotic EED of $\mathbf{R}_s\mathbf{R}_n^{-1}$ in the point-to-point MIMO case [7]. Note that for these conclusions, we require that the limits can be taken inside the integral in (1). A corresponding proof is skipped due to space constraints.

IV. Asymptotic Analysis

This section provides the proof of the theorem. We start out with the following Lemma:

Lemma 1. *Let $\varepsilon > 0$ and g be some function $g: \mathbb{R} \longrightarrow \mathbb{N} : \kappa \longrightarrow g(\kappa)$.*

- *Then, for all $c \in \mathbb{C}$ and $g(\kappa) \in \mathcal{O}(\kappa^{1-\varepsilon})$*
$$\lim_{\kappa\to\infty}\left(\frac{c}{\kappa}+1\right)^{g(\kappa)} = 1.$$
- *Then, for all negative c and $g(\kappa) \in \Omega(\kappa^{1+\varepsilon})$,*
$$\lim_{\kappa\to\infty}\left(\frac{c}{\kappa}+1\right)^{g(\kappa)} = 0.$$
- *Then, for all $c \in \mathbb{C}$ and $g(\kappa) \in \Theta(\kappa)$ there exist finite constants M_1 and M_2, $M_2 \geq M_1 > 0$, such that*
$$\liminf_{\kappa\to\infty}\left|\left(\frac{c}{\kappa}+1\right)^{g(\kappa)}\right| \geq e^{|c|M_1},$$
$$\limsup_{\kappa\to\infty}\left|\left(\frac{c}{\kappa}+1\right)^{g(\kappa)}\right| \leq e^{|c|M_2}.$$

Proof. The lemma follows from the fact that the limit can be taken inside a continuous function, which allows us to write

$$\begin{aligned}\lim_{\kappa\to\infty}\left(\frac{c}{\kappa}+1\right)^{g(\kappa)} &= \exp\left(\lim_{\kappa\to\infty}\mathrm{Log}\left(\frac{c}{\kappa}+1\right)^{g(\kappa)}\right) \\ &= \exp\left(\lim_{\kappa\to\infty} g(\kappa)\log\left|\frac{c}{\kappa}+1\right| + \imath\arg\left\{\frac{c}{\kappa}+1\right\}\right) \\ &= \exp\left(\lim_{\kappa\to\infty} g(\kappa)\log\left|\frac{c}{\kappa}+1\right|\right) \\ &= \exp\left(\lim_{\kappa\to\infty}\Re\left\{g(\kappa)\mathrm{Log}\left(\frac{c}{\kappa}+1\right)\right\}\right). \end{aligned} \quad (2)$$

From the rule of Bernoulli-l'Hospital, we know that

$$\lim_{\kappa\to\infty} M\kappa^{\gamma}\cdot \mathrm{Log}\left(\frac{c}{\kappa}+1\right) = \lim_{\kappa\to\infty}\frac{cM\kappa^{\gamma}}{\gamma\,(c+\kappa)}, \tag{3}$$

where γ and M are positive constants.

If $g(\kappa)\in\mathcal{O}(\kappa^{1-\varepsilon})$, by definition there exists some $M>0$, such that the absolute value of the argument of the exponential function in (2) can be upper-bounded by

$$\lim_{\kappa\to\infty}\left|g(\kappa)\cdot \mathrm{Log}\left(\frac{c}{\kappa}+1\right)\right| \le \lim_{\kappa\to\infty}\left|M\kappa^{1-\varepsilon}\cdot \mathrm{Log}\left(\frac{c}{\kappa}+1\right)\right|. \tag{4}$$

Evaluating (3) for $\gamma = 1-\varepsilon$ renders this upper bound zero, which establishes that also the left hand side (LHS) of (4) becomes zero and (2) evaluates to one in this case.

Analogously, if $g(\kappa)\in\Omega(\kappa^{1+\varepsilon})$, there exists some $M>0$, such that the argument of the exponential function in (2) can be upper bounded according to (note that κ and $g(\kappa)$ are positive, while the logarithm is negative due to the negative c)

$$\lim_{\kappa\to\infty} g(\kappa)\cdot\log\left(\frac{c}{\kappa}+1\right) \le \lim_{\kappa\to\infty} M\kappa^{1+\varepsilon}\cdot\log\left(\frac{c}{\kappa}+1\right). \tag{5}$$

The limit (3) does not exist for $\gamma = 1+\varepsilon$ implying that both sides of (5) go to minus infinity, and (2) to zero in turn.

Finally, if $g(\kappa)\in\Theta(\kappa)$, there exist constants M_1 and M_2, $M_2\ge M_1>0$, and κ_0, such that for all $\kappa\ge\kappa_0$, according to (3) evaluated for $\gamma=1$ the exponent in (2) is sandwiched between

$$\begin{aligned} |c|M_1 &= \lim_{\kappa\to\infty}\left|M_1\kappa\cdot \mathrm{Log}\left(\frac{c}{\kappa}+1\right)\right| \\ &\le \left|g(\kappa)\cdot \mathrm{Log}\left(\frac{c}{\kappa}+1\right)\right| \\ &\le \lim_{\kappa\to\infty}\left|M_2\kappa\cdot \mathrm{Log}\left(\frac{c}{\kappa}+1\right)\right| = |c|M_2, \end{aligned}$$

which completes the proof of the lemma. ■

In the following propositions, we characterize the *individual* asymptotic EEDs of $\mathbf{R}_s$ and $\mathbf{R}_n$ in the limit $\beta_{\mathcal{R}}\to\infty$ for the different couplings between $\beta_{\mathcal{R}}$ and L. Rather than $\mathbf{R}_s$ and $\mathbf{R}_n$, we consider the normalized matrices

$$\tilde{\mathbf{R}}_s = P^{-1}\alpha^{-L}\cdot\mathbf{R}_s \quad\text{and}\quad \tilde{\mathbf{R}}_n = \frac{1-\alpha}{1-\alpha^{L+1}}\cdot\mathbf{R}_n.$$

Note that due to the particular choice of α and P in the theorem, we have the relation $\mathbf{R}_s\mathbf{R}_n^{-1} = \mathrm{snr}\cdot\tilde{\mathbf{R}}_s\tilde{\mathbf{R}}_n^{-1}$.

Proposition 1. *Given the assumptions of the theorem, the asymptotic EED of $\tilde{\mathbf{R}}_s$ in the limit $\beta_{\mathcal{R}}\to\infty$ converges pointwise to*

$$F_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(x) = \begin{cases} F_{\mathrm{MP}}^{(\beta_{\mathcal{S}})}(x), & \text{if}\quad L\in\mathcal{O}(\beta_{\mathcal{R}}^{1-\varepsilon}),\\ \sigma(x), & \text{if}\quad L\in\Omega(\beta_{\mathcal{R}}^{1+\varepsilon}).\end{cases}$$

Also, if $L\in\Theta(\beta_{\mathcal{R}})$ the asymptotic EED of $\tilde{\mathbf{R}}_s$ converges pointwise to a distribution function $F_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(x)\neq\sigma(x)$.

Proof. An implicit equation for the Stieltjes transform[2] of the asymptotic EED of $\tilde{\mathbf{R}}_s$ is provided in [1]. Adapted to our notation it is given by

$$\overbrace{\left(\frac{sG_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{R}},L,\beta_{\mathcal{S}})}(s)-1}{\beta_{\mathcal{R}}}+1\right)^{L-1}}^{\Psi_s} \times\frac{G_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{R}},L,\beta_{\mathcal{S}})}(s)}{\beta_{\mathcal{S}}}\cdot\frac{sG_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{R}},L,\beta_{\mathcal{S}})}(s)-1+\beta_{\mathcal{S}}}{\beta_{\mathcal{R}}} \times\left(sG_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{R}},L,\beta_{\mathcal{S}})}(s)-1+\beta_{\mathcal{R}}\right)+sG_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{R}},L,\beta_{\mathcal{S}})}(s)=1. \tag{6}$$

We apply Lemma 1 to Ψ_s, where we identify $\beta_{\mathcal{R}}$ with κ and L with $g(\kappa)$. In the limit $\beta_{\mathcal{R}}\to\infty$ the implicit equation (6) simplifies as follows:

1) If $L\in\mathcal{O}(\beta_{\mathcal{R}}^{1-\varepsilon})$, then $\Psi_s\to 1$, and thus
$$\beta_{\mathcal{S}}^{-1}sG_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})^2}(s)+\left(s+1-\beta_{\mathcal{S}}^{-1}\right)G_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(s)=1. \tag{7}$$
2) If $L\in\Omega(\beta_{\mathcal{R}}^{1+\varepsilon})$, then $\Psi_s\to 0$, and thus
$$G_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}=\frac{1}{s}, \tag{8}$$
3) If $L\in\Theta(\beta_{\mathcal{R}})$, then $\Psi_s = e^{d(s,\beta_{\mathcal{R}},L)(sG_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(s)-1)}$ with $0<|d(s,\beta_{\mathcal{R}},L)|<\infty$ for all s, $\beta_{\mathcal{R}}$ and L, and thus
$$G_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(s)\Bigg(e^{d(s,\beta_{\mathcal{R}},L)(sG_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(s)-1)} \times\left(\beta_{\mathcal{S}}^{-1}sG_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(s)+1-\beta_{\mathcal{S}}^{-1}\right)+s\Bigg)=1. \tag{9}$$

The solution to (7) is the Stieltjes transform of the Marčenko-Pastur law with parameter $\beta_{\mathcal{S}}$. For (8), we used that $sG_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(s)$ must be smaller than one, which is required for the application of Lemma 1: this is indeed the case, since $G(s)$ is positive for positive s, and thus the LHS of (6) would be larger than one otherwise (cf. proof of [1, Theorem 4]). Finally, there exists no closed form solution to the implicit equation (9). However, it is easily verified that $G_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(s)=s^{-1}$ does not satisfy (9) for any function $d(s,\beta_{\mathcal{R}},L)$ taking on finite values only. ■

Proposition 2. *For $L\in\mathcal{O}(\beta_{\mathcal{R}}^{1-\varepsilon})$ and $\beta_{\mathcal{R}}\to\infty$, the asymptotic EED of $\tilde{\mathbf{R}}_n^{-1}$ converges pointwise to*

$$F_{\tilde{\mathbf{R}}_n^{-1}}(x)=\sigma(x-1).$$

Moreover, for arbitrary growth of L with $\beta_{\mathcal{R}}$ the number of eigenvalues of $\tilde{\mathbf{R}}_n$ that go to infinity in the limit $n_{\mathcal{D}}\to\infty$ is in $\mathcal{O}(n_{\mathcal{D}}^{1-\varepsilon})$.

The proof relies on the following Lemma:

Lemma 2. *Let $\mathbf{A}(\gamma)\in\mathbb{C}^{n\times n}$ be a sequence of positive semidefinite random matrices with asymptotic EED $F_{\mathbf{A}}^{(\gamma)}(x)$ that fulfills $\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}[\mathbf{A}(\gamma)] = 1$ and $\lim_{n\to\infty}\lambda_{\max}\{\mathbf{A}(\gamma)\}<\infty$ for all γ. The following types of convergence are equivalent ($\|\cdot\|_{\mathrm{Tr}}$ denotes the trace norm):*

1) $\lim_{\gamma\to\infty}\lim_{n\to\infty}\frac{1}{n}\|\mathbf{I}_n-\mathbf{A}(\gamma)\|_{\mathrm{Tr}}=0$ *almost surely.*
2) $\lim_{\gamma\to\infty}|F_{\mathbf{A}}^{(\gamma)}(x)-\sigma(x-1)|=0$ *for all x.*
3) $\lim_{\gamma\to\infty}\int_0^{\infty}|F_{\mathbf{A}}^{(\gamma)}(x)-\sigma(x-1)|\cdot dx=0.$

[2] We use the definition of the Stieltjes transform $G(s)\triangleq\int(x+s)^{-1}dF(x)$.

We skip the proof of the lemma due to space constraints and continue with the proof Proposition 2:

Proof. Let us denote the lth summand of $\mathbf{R}_n$ by $\mathbf{R}_{n,l}$. The implicit equation for the Stieltjes transform of the asymptotic EED of $\mathbf{R}_{n,l}$, where $l \in \{1,\ldots,L\}$, is given by [1]

$$\frac{G^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(s)}{\beta_\mathcal{R}} \cdot \overbrace{\left(\frac{sG^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(s) - 1 + \beta_\mathcal{R}}{\beta_\mathcal{R}}\right)^{l-1}}^{\Psi_{n,l}} \times \left(sG^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(s) - 1 + \beta_\mathcal{R}\right) + sG^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(s) = 1. \tag{10}$$

If $l \in \mathcal{O}\left(\beta_\mathcal{R}^{1-\varepsilon}\right)$, $\Psi_{n,l}$ converges to one in the limit $\beta_\mathcal{R} \to \infty$ by Lemma 1. Thus, (10) simplifies to $G_{\mathbf{R}_{n,l}}(s) = (s+1)^{-1}$. Let us define $\gamma \triangleq \beta_\mathcal{R}/l$ and rewrite this as

$$\lim_{\gamma\to\infty} G^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(s) = (s+1)^{-1}. \tag{11}$$

We have thus shown that for each $\delta > 0$ and each s there exists some $\gamma^{(0)}(s)$ such that for all $\gamma > \gamma^{(0)}(s)$, we have $|G^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(s) - (s+1)^{-1}| < \delta$. Next, we show that for each $\delta > 0$ and each s there exists some $\gamma^{(0)}(s)$ such that for all $\gamma > \gamma^{(0)}(s)$, we have

$$\max_{l\in\{1,\ldots,L\}} \left|G^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(s) - (s+1)^{-1}\right| < \delta. \tag{12}$$

This means the L terms are uniformly bounded as $\beta_\mathcal{R}$ (and possibly L) tends to infinity. This follows, since whenever

$$\left|G^{(\beta_\mathcal{R},L)}_{\mathbf{R}_{n,L}}(s) - (s+1)^{-1}\right| < \delta \text{ for all } \gamma > \gamma^{(0)}(s),$$

inequaility (12) must be fulfilled (γ is minimized for $l = L$).

In a next step, we conclude that also $F_{\tilde{\mathbf{R}}}(x) = \sigma(x-1)$, if $L \in \mathcal{O}\left(\beta_\mathcal{R}^{1-\varepsilon}\right)$. We do this by applying Lemma 2, which allows to consider the trace norm of the difference between $\tilde{\mathbf{R}}_n$ and the identity matrix (without verifying that $\tilde{\mathbf{R}}_n$ fulfills the assumptions of Lemma 2 due to space constraints). We obtain that given any $\varepsilon > 0$ there are $\beta^{(0)}_\mathcal{R}$ and $n^{(0)}_\mathcal{D}(\beta_\mathcal{R})$ such that for all $\beta_\mathcal{R} \geq \beta^{(0)}_\mathcal{R}$ and all $n_\mathcal{D} \geq n^{(0)}_\mathcal{D}(\beta_\mathcal{R})$ almost surely

$$\begin{aligned}
&\frac{1}{n_\mathcal{D}}\left\|\mathbf{I}_{n_\mathcal{D}} - \tilde{\mathbf{R}}_n\right\|_{\mathrm{Tr}} = \frac{1}{n_\mathcal{D}}\left\|\mathbf{I}_{n_\mathcal{D}} - \frac{1-\alpha}{1-\alpha^{L+1}}\cdot\mathbf{R}_n\right\|_{\mathrm{Tr}} \\
&= \frac{1}{n_\mathcal{D}}\left\|\frac{1-\alpha}{1-\alpha^{L+1}}\sum_{l=0}^{L}\left(\alpha^l\cdot\mathbf{I}_{n_\mathcal{D}} - \mathbf{R}_{n,l}\right)\right\|_{\mathrm{Tr}} \\
&= \frac{1}{n_\mathcal{D}}\left\|\frac{1-\alpha}{1-\alpha^{L+1}}\sum_{l=0}^{L}\alpha^l\left(\mathbf{I}_{n_\mathcal{D}} - \frac{1}{n^l_\mathcal{R}}\mathbf{H}_1\cdots\mathbf{H}_l\mathbf{H}^H_l\cdots\mathbf{H}^H_1\right)\right\|_{\mathrm{Tr}} \\
&\leq \frac{1-\alpha}{1-\alpha^{L+1}}\sum_{l=0}^{L}\frac{\alpha^l}{n_\mathcal{D}}\left\|\mathbf{I}_{n_\mathcal{D}} - \frac{1}{n^l_\mathcal{R}}\mathbf{H}_1\cdots\mathbf{H}_l\mathbf{H}^H_l\cdots\mathbf{H}^H_1\right\|_{\mathrm{Tr}} \qquad (13) \\
&= \frac{1-\alpha}{1-\alpha^{L+1}}\sum_{l=0}^{L}\alpha^l\int_0^\infty\left|F^{(n_\mathcal{D},\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(x) - \sigma(x-1)\right|dx \qquad (14) \\
&\leq \frac{1-\alpha}{1-\alpha^{L+1}}\sum_{l=0}^{L}\alpha^l\Bigg(\int_0^\infty\left|F^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(x) - \sigma(x-1)\right|dx \\
&\qquad + \int_0^\infty\left|F^{(n_\mathcal{D},\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(x) - F^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(x)\right|dx\Bigg) \qquad (15) \\
&\leq \max_{l\in\{1,\ldots,L\}}\left\{\int_0^\infty\left|F^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(x) - \sigma(x-1)\right|dx\right\} \\
&\qquad + \max_{l\in\{1,\ldots,L\}}\left\{\int_0^\infty\left|F^{(n_\mathcal{D},\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(x) - F^{(\beta_\mathcal{R},l)}_{\mathbf{R}_{n,l}}(x)\right|dx\right\} \qquad (16) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We obtain (13) by applying the triangle inequality and using the homogeneity of the trace norm. Equality between (13) and (14) is established by the following chain of identities for a positive semidefinite matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ (σ_i and λ_i denote ith singular- and eigenvalue, respectively):

$$\begin{aligned}
&n^{-1}\|\mathbf{I}_n - \mathbf{A}(\gamma)\|_{\mathrm{Tr}} = n^{-1}\sum_{i=1}^{n}\sigma_i\{\mathbf{I}_n - \mathbf{A}(\gamma)\} \\
&= n^{-1}\sum_{i=1}^{n}|\lambda_i\{\mathbf{I}_n - \mathbf{A}(\gamma)\}| = n^{-1}\sum_{i=1}^{n}|1 - \lambda_i\{\mathbf{A}(\gamma)\}| \\
&= n^{-1}\sum_{i:\lambda_i\{\mathbf{A}(\gamma)\}\leq 1}(1 - \lambda_i\{\mathbf{A}(\gamma)\}) + n^{-1}\sum_{i:\lambda_i\{\mathbf{A}(\gamma)\}>1}(\lambda_i\{\mathbf{A}(\gamma)\} - 1) \\
&= \int_0^1|F^{(n,\gamma)}_{\mathbf{A}}(x)|dx + \int_1^\infty|F^{(n,\gamma)}_{\mathbf{A}}(x) - 1|dx \\
&= \int_0^\infty|F^{(n,\gamma)}_{\mathbf{A}}(x) - \sigma(x-1)|dx.
\end{aligned}$$

Eq. (15) follows by adding and substracting $F^{(\beta_\mathcal{R})}_{\mathbf{R}_{n,l}}(x)$ and repeated application of the triangle inequality. In (16) we upper-bound the individual integrals by the largest ones. In the final step, we upper bound both terms in (16) by $\varepsilon/2$. For the first term, we can fix a $\beta^{(0)}_\mathcal{R}$ such that this upper-bound holds for all $\beta_\mathcal{R} \geq \beta^{(0)}_\mathcal{R}$ by (11) and Lemma 2. Having fixed $\beta_\mathcal{R}$ (and thus L), we can finally choose an $n^{(0)}_\mathcal{D}(\beta_\mathcal{R})$ large enough such that for all $n_\mathcal{D} \geq n^{(0)}_\mathcal{D}(\beta_\mathcal{R})$ the second term is smaller than $\varepsilon/2$ (since $\lim_{n_\mathcal{D}\to\infty}\lambda_{\max}\{\tilde{\mathbf{R}}_n\} < \infty$ for fixed L, the limit can be taken inside the integral due to the uniform convergence of $F^{(n_\mathcal{D},\beta_\mathcal{R},l)}_{\tilde{\mathbf{R}}_n}$ to $F^{(\beta_\mathcal{R},l)}_{\tilde{\mathbf{R}}_n}$).

We have thus shown that $F^{(\beta_\mathcal{R})}_{\tilde{\mathbf{R}}_n}(x)$ converges pointwise to $\sigma(x-1)$ as $\beta_\mathcal{R} \to \infty$. Since the eigenvalues of the inverse of $\tilde{\mathbf{R}}_n$ are the inverse eigenvalues of $\tilde{\mathbf{R}}_n$, i.e., $\lambda_k\{\tilde{\mathbf{R}}^{-1}_n\} = \lambda^{-1}_k\{\tilde{\mathbf{R}}_n\}$, we conclude that also $F^{(\beta_\mathcal{R},L)}_{\tilde{\mathbf{R}}^{-1}_n}(x)$ converges pointwise to $\sigma(x-1)$.

The second part of the proposition is established as follows. For *arbitrarily large* $\beta_\mathcal{R}$ and L, the number of eigenvalues of $\tilde{\mathbf{R}}_n$ that go to infinity as $n_\mathcal{D} \to \infty$ must be in $\mathcal{O}(n^{1-\epsilon}_\mathcal{D})$ for some $\epsilon > 0$, since a linear growth would contradict

$$\begin{aligned}
&\lim_{n_\mathcal{D}\to\infty}\frac{1}{n_\mathcal{D}}\sum_{i=1}^{n_\mathcal{D}}\lambda_i\left\{\tilde{\mathbf{R}}_n\right\} = \lim_{n_\mathcal{D}\to\infty}\frac{1}{n_\mathcal{D}}\sum_{i=1}^{n_\mathcal{D}}\lambda_i\left\{\frac{1-\alpha}{1-\alpha^{L+1}}\mathbf{R}_n\right\} \\
&= \lim_{n_\mathcal{D}\to\infty}\frac{1}{n_\mathcal{D}}\mathrm{Tr}\left[\frac{1-\alpha}{1-\alpha^{L+1}}\cdot\mathbf{R}_n\right]
\end{aligned}$$

$$= \frac{1-\alpha}{1-\alpha^{L+1}} \sum_{l=0}^{L} \alpha^l \lim_{n_{\mathcal{D}}\to\infty} \frac{1}{n_{\mathcal{D}}} \mathrm{Tr}\left[\frac{1}{n_{\mathcal{R}}^l} \mathbf{H}_1 \cdots \mathbf{H}_l \mathbf{H}_l^H \cdots \mathbf{H}_1^H\right]$$
$$= \frac{1-\alpha}{1-\alpha^{L+1}} \sum_{l=0}^{L} \alpha^l = 1.$$

The evaluation of the limit follows due to the asymptotic freeness of the channel matrices inside the trace. Since $\lambda_i\{\tilde{\mathbf{R}}_n^{-1}\} = \lambda_i^{-1}\{\tilde{\mathbf{R}}_n\}$, we conclude that also the number of eigenvalues of $\tilde{\mathbf{R}}_n^{-1}$ that go to zero must be $\mathcal{O}(n_{\mathcal{D}}^{1-\varepsilon})$. ■

With Propositions 1 & 2 ready to hand, we can now prove the theorem.

Proof of Theorem. We go through the different scaling behaviors of L with $\beta_{\mathcal{R}}$ in the following. For the sake of brevity we restrict ourselves to conceptual arguments rather than giving the proof in full detail.

Case $L \in \Omega\left(\beta_{\mathcal{R}}^{1+\varepsilon}\right)$: This case follows immediately from Proposition 1. Since asymptotically all but less than linearly many eigenvalues of $\tilde{\mathbf{R}}_s$ vanish, also all but less than linearly many eigenvalues of $\mathrm{snr}\cdot\tilde{\mathbf{R}}_s\tilde{\mathbf{R}}_n^{-1}$ must approach zero. This is since the rank deficiency of $\tilde{\mathbf{R}}_s$ cannot be compensated for by a multiplication with any other matrix.

Case $L \in \Theta\left(\beta_{\mathcal{R}}\right)$: We can argue based on Sylvester's inequality that

$$\mathrm{rk}\left\{\tilde{\mathbf{R}}_s\tilde{\mathbf{R}}_n^{-1}\right\} \geq \mathrm{rk}\left\{\tilde{\mathbf{R}}_s\right\} + \mathrm{rk}\left\{\tilde{\mathbf{R}}_n^{-1}\right\} - n_{\mathcal{D}} \in \Theta(n_{\mathcal{D}}).$$

Here, we use that a nonzero fraction of the eigenvalues of $\tilde{\mathbf{R}}_s$ remains nonzero as $\beta_{\mathcal{R}} \to \infty$ by Proposition 1, or equivalently $\mathrm{rk}\{\tilde{\mathbf{R}}_s\} \in \Theta(n_{\mathcal{D}})$, while $\mathrm{rk}\{\tilde{\mathbf{R}}_n^{-1}\} - n_{\mathcal{D}} \in \mathcal{O}(n_{\mathcal{D}}^{1-\epsilon})$ for some $\epsilon > 0$ by Proposition 2.

Case $L \in \mathcal{O}\left(\beta_{\mathcal{R}}^{1-\varepsilon}\right)$: Since $F_{\tilde{\mathbf{R}}_n^{-1}}^{(\beta_{\mathcal{S}})}(x) = \sigma(x-1)$, we can write

$$F_{\mathrm{snr}\tilde{\mathbf{R}}_s\tilde{\mathbf{R}}_n^{-1}}^{(\beta_{\mathcal{S}})}(x) = F_{\mathrm{snr}\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(x) = F_{\mathrm{snr}\mathbf{R}_s}^{(\beta_{\mathcal{S}})}(x) = F_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}\left(\frac{x}{\mathrm{snr}}\right).$$

By Proposition 1, $F_{\tilde{\mathbf{R}}_s}^{(\beta_{\mathcal{S}})}(\cdot)$ obeys the Marčenko-Pastur law for $L \in \mathcal{O}(\beta_{\mathcal{R}}^{1-\varepsilon})$, which establishes this case. ■

V. Computer Simulations

We support our asymptotic result by means of computer simulations. We specify the distribution of the elements of the $\mathbf{H}_l$ as circularly symmetric complex Gaussian. We fix $n_{\mathcal{S}} = n_{\mathcal{D}} = 10$ and $\mathrm{snr} = 10$ dB. We plot the normalized ergodic sum-capacity $C_0 = L_0 \cdot C$ as obtained through Monte Carlo simulations versus the number of relay clusters, L. The number of relays per cluster evolves with the number of clusters according to $n_{\mathcal{R}} = L^{\gamma}$. The upper plot in Fig. 2 shows the case of linear and slower scaling of L with $n_{\mathcal{R}}$. For $\gamma = 1$ the curve converges to some constant which is smaller than the point-to-point capacity, but nonzero. For $\gamma > 1$, we observe that the point-to-point limit is approached the faster, the larger γ is chosen. The lower plot in Fig. 2 shows the case of faster than linear scaling of L with $n_{\mathcal{R}}$. While C_0 decreases rapidly for constant relay numbers ($\gamma = 0$), it turns out that a moderate growth of $n_{\mathcal{R}}$ with L suffices to slow down the capacity decay significantly.

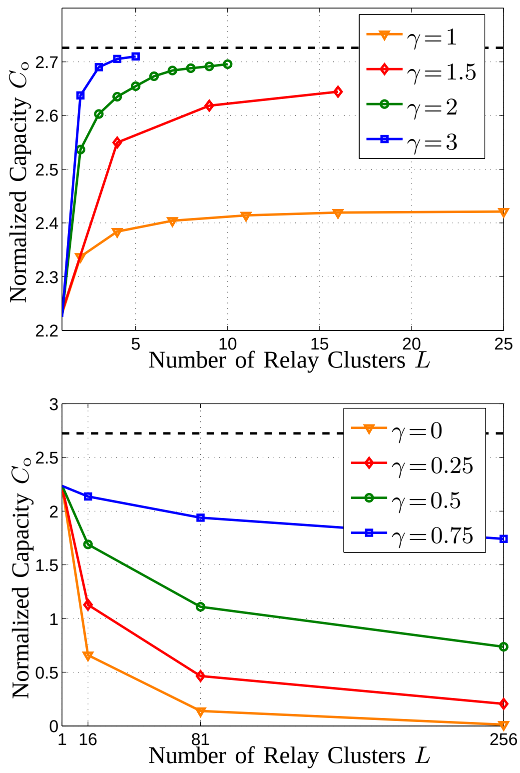

Fig. 2. Normalized capacity vs. L. Dashed: point-to-point MIMO capacity.

VI. Conclusion

We provided a criterion on the growth of the number of relays per stage with the number of hops in an amplify-and-forward multihop network that determines whether or not sum-capacity scales linearly in $\min\{n_{\mathcal{S}}, n_{\mathcal{D}}\}$ in the limit of infinitely many hops.

VII. Acknowledgment

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