On the Asymptotic Capacity of the Rayleigh Fading Amplify-and-Forward MIMO Relay Channel

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***Abstract*— In this paper the ergodic mutual information of the i.i.d. Rayleigh fading amplify-and-forward MIMO relay channel without direct link between source and destination is derived in the large array limit. By means of the *replica method* an expression is obtained, whose parameters are determined by a nonlinear equation system. This system can be solved in closed form. Comparisons with Monte Carlo simulations show that the asymptotic expression serves as an excellent approximation for systems with only very few antennas at each node.**

I. Introduction

Cooperative relaying has obtained major attention in the wireless communications community in recent years due to its various potentials regarding the enhancement of diversity, achievable rates and range. An important milestone within the wide scope of this field is the understanding of the fundamental limits of the *MIMO relay channel*. Such a channel consists of a source, relay and destination terminal, each equipped with multiple antennas.

Generally, there are different ways of including relays in the transmission between a source and a destination terminal. Most commonly relays are introduced to either decode the noisy signal from the source or another relay, to re-encode the information bits and to transmit it to another relay (*multi-hop*) or the destination terminal (*two-hop*). Or the relay simply forwards a linearly modified version of the noisy signal. These relaying strategies are referred to as decode-and forward (DF) and amplify-and-forward (AF), respectively. Some advantages of the AF approach addressed in this paper are its independence of the modulation format at the source, the avoidance of decoding delay and the availability of full distributed diversity [1]. Another approach is the so called compress-and-forward strategy (CF), which quantizes the received signal and re-encodes the resulting samples efficiently.

We briefly give an overview over important contributions to the field of cooperative communications and relaying. The capability of relays to provide diversity for combating multi-path fading has been studied in [1] and [2]. In [3] the potential of spatial multiplexing gain enhancement in correlated fading channels by means of relays has been demonstrated. Tight upper and lower bounds on the capacity of the fading relay channel are provided in [4] and [5]. Furthermore, in [6] the capacity has been shown to scale like $N/2 \log K$ in a fading two-hop relay network with a single source and a single destination equipped with N antennas each and K distributed relays, if all the relays dispose of full channel state information and operate in a half-duplex fashion.

In this paper we focus on the i.i.d. Rayleigh fading two-hop AF MIMO relay channel without direct link between source and destination. This channel is of particular interest in the case that some of the relay stage antennas are not collocated such that that the DF strategy is rendered interference limited. Our quantity of interest is the mean of the mutual information of this channel, which completely determines the achievable rate in a fast fading communication channel. Seeking for closed form expressions of the mean mutual information in MIMO systems is for most channel models a very difficult task. For the conventional point-to-point MIMO channel it therefore turned out to be useful to defer the analysis to the regime of large antenna numbers. For the i.i.d. Rayleigh fading MIMO channel a closed form expression was obtained in [7]. The result is obtained via the deterministic asymptotic eigenvalue spectrum of the matrix inside the capacity $\log\det$-formula. Generally, such large array results turned out to be very tight approximations of the respective quantities in finite dimensional systems.

The mean mutual information of Rayleigh fading amplify-and-forward MIMO relay channels in the large array limit has also been studied in [8] for the special case of a forwarding matrix proportional to the identity matrix. In this paper a fourth order equation for the Stieltjes transform of the corresponding asymptotic eigenvalue spectrum is found, which allows for a numerical evaluation of the mutual information. Analyzing this scenario by means of the replica method yields a closed form expression instead. Moreover, our approach allows for the treatment of correlated fading and the evaluation of higher moments of the mutual information.

The key tool enabling the large array analysis in this paper is the so called replica method. It was introduced by Edwards and Anderson in [9] and has its origins in physics where it is applied to large random systems, as they arise, e.g., in statistical mechanics. In the context of channel capacity it was applied by Tanaka in [10] for the first time. Moustakas et al. [11] finally used a framework utilizing the replica method developed in [12] to evaluate the cumulant moments of the mutual information of the Rayleigh fading MIMO channel in the presence of correlated MIMO interference. Though not being proven in a rigorous way yet, the replica method is a particularly attractive tool in large random matrix theory,

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since it allows for the evaluation of arbitrary moments. Free probability theory, e.g., is restricted to the mean [13].

II. Channel Model and Mutual Information

The two-hop amplify-and-forward MIMO relay channel under consideration is defined as follows. A source, a relay and a destination terminal are equipped with n antennas each. We allow for communication from source to relay and from relay to destination. Particularly, we do not assume a direct communication link between source and destination. Both the first hop from source to relay and the second hop from relay to destination are modeled as frequency-flat, i.e., the transmit symbol duration is much longer than the maximum delay spread of the first hop and second hop MIMO channel. We denote the channel matrix of the first hop by $\mathbf{H}_1 \in \mathbb{C}^{n\times n}$, the one of the second hop by $\mathbf{H}_2 \in \mathbb{C}^{n\times n}$. Their entries are assumed to be zero mean circular symmetric complex Gaussian (ZMCSCG) random variables of unit variance. Furthermore, we assume that the relay simply forwards scaled versions of its received signals, which corresponds to a forwarding matrix $\mathbf{F} = \sqrt{\alpha/n}\cdot\mathbf{I}_n$, where α corresponds to the overall power gain of the relay terminal. With s the transmit symbol vector and $\mathbf{n}_\mathrm{n}$ and $\mathbf{n}_\mathrm{d}$ the relay and destination noise vectors respectively, the end-to-end input-output-relation of this channel is then given by

$$\mathbf{y} = \mathbf{H}_2\mathbf{F}\mathbf{H}_1\mathbf{s} + \mathbf{H}_2\mathbf{F}\mathbf{n}_\mathrm{r} + \mathbf{n}_\mathrm{d}. \tag{1}$$

We assume all channel matrix elements to be constant during a certain interval and to change independently from interval to interval (block fading). The input symbols are chosen to be i.i.d. ZMCSCGs with variance ρ, i.e., $\mathsf{E}[\mathbf{s}\mathbf{s}^\mathrm{H}] = \rho/n\mathbf{I}_n$, the additive noise at relay and destination is assumed to be white in both space and time and is modeled as ZMCSCG with unit variance, i.e., $\mathsf{E}[\mathbf{n}_\mathrm{r}\mathbf{n}_\mathrm{r}^\mathrm{H}] = \mathbf{I}_\mathrm{r}$ and $\mathsf{E}[\mathbf{n}_\mathrm{d}\mathbf{n}_\mathrm{d}^\mathrm{H}] = \mathbf{I}_n$.

With this notation the mutual information[1] conditioned on $\mathbf{H}_1$ and $\mathbf{H}_2$ in nats per channel use and spatial dimension can be written as

$$I = \frac{1}{n}\ln\frac{\det\left(\mathbf{I}_n + \frac{\alpha}{n}\mathbf{H}_2\mathbf{H}_2^\mathrm{H} + \frac{\rho\alpha}{n^2}\mathbf{H}_2\mathbf{H}_1\mathbf{H}_1^\mathrm{H}\mathbf{H}_2^\mathrm{H}\right)}{\det\left(\mathbf{I}_n + \frac{\alpha}{n}\mathbf{H}_2\mathbf{H}_2^\mathrm{H}\right)}, \tag{2}$$

where

$$\mathbf{I}_n + \frac{\alpha}{n}\mathbf{H}_2\mathbf{H}_2^\mathrm{H} \tag{3}$$

corresponds to the overall noise covariance matrix at destination and

$$\frac{\rho\alpha}{n^2}\mathbf{H}_2\mathbf{H}_1\mathbf{H}_1^\mathrm{H}\mathbf{H}_2^\mathrm{H} \tag{4}$$

corresponds to the signal plus noise covariance matrix at the destination.

Due to the randomness in $\mathbf{H}_1$ and $\mathbf{H}_2$ also I is a random variable.

[1] In this chapter we pass on the common pre-log factor $1/2$, which accounts for the use of two time slots necessary in half-duplex relay protocols.

III. Main Result

The main result of this paper is summarized in the following theorem:

Theorem. *For the mutual information I as defined in (2) the asymptotic mean is given by*

$$\begin{aligned}\lim_{n\to\infty}\mathsf{E}[I] \;=\;& \ln(1+\rho r_1) + \ln(1+\alpha r_2)\\ &+\ln(1+q_2+q_1q_2)\\ &-\ln(1+\alpha t) - \ln(1+s)\\ &-(r_1q_1 + r_2q_2 - st)\end{aligned} \tag{5}$$

where the coefficients are determined by the nonlinear equation system

$$q_1 = \frac{\rho}{1+\rho r_1} \qquad r_1 = \frac{q_2}{1+q_1q_2+q_2} \tag{6}$$

$$q_2 = \frac{\alpha}{1+\alpha r_2} \qquad r_2 = \frac{1+q_1}{1+q_1q_2+q_2} \tag{7}$$

$$s = \frac{\alpha}{1+\alpha t} \qquad t = \frac{1}{1+s}. \tag{8}$$

The relevant solution to this system is the (unique) one for which all coefficients are real valued and positive.

Note that this system of equations can be solved in *closed form.*

IV. Proof

A. Integral Identities

We will need two useful integral identities in the subsequent proof. Before stating them we introduce a compact notation for products of differentials arising when integration over elements of matrices is performed. With $\imath = \sqrt{-1}$ as well as $\Re\{z\}$ and $\Im\{z\}$ the real and imaginary part of a complex variable z, we introduce the following integral measures using the same notation as [11]:

- For $\mathbf{X},\mathbf{Y} \in \mathbb{C}^{m\times n}$ and x_{kl}, y_{kl} the respective matrix elements we define

$$d\mathbf{X} \triangleq \prod_{k=1}^{m}\prod_{l=1}^{n}\frac{d\Re\{x_{kl}\}d\Im\{x_{kl}\}}{2\pi}, \tag{9}$$

- For $\mathbf{X},\mathbf{Y} \in \mathbb{C}^{m\times m}$ and x_{kl}, y_{kl} the respective matrix elements we define

$$d\mu(\mathbf{X},\mathbf{Y}) \triangleq \prod_{k=1}^{m}\prod_{l=1}^{m}\frac{dx_{kl}dy_{lk}}{2\pi\imath}. \tag{10}$$

With this notation as well as $\otimes$ the Kronecker product operator we specify the following identities, which are all proven in [11]:

- For $\mathbf{M} \in \mathbb{C}^{n\times n}$, $\mathbf{N} \in \mathbb{C}^{\nu\times\nu}$ positive definite, and $\mathbf{X},\mathbf{A},\mathbf{B} \in \mathbb{C}^{n\times\nu}$ we have

$$\int \exp\left(-\frac{1}{2}\mathrm{Tr}\left(\mathbf{N}\mathbf{X}^\mathrm{H}\mathbf{M}\mathbf{X} + \mathbf{A}^\mathrm{H}\mathbf{X} - \mathbf{X}^\mathrm{H}\mathbf{B}\right)\right)d\mathbf{X} = \frac{\exp\left(-\frac{1}{2}\mathrm{Tr}\left(\mathbf{N}^{-1}\mathbf{A}^\mathrm{H}\mathbf{M}^{-1}\mathbf{B}\right)\right)}{\det\left(\mathbf{N}\otimes\mathbf{M}\right)}. \tag{11}$$

- For $\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B} \in \mathbb{C}^{\nu \times \nu}$ and the integration of the elements of $\mathbf{X}$ and $\mathbf{Y}$ along contours in the complex space parallel to the real and imaginary axis we have

$$\int \exp\left(\mathrm{Tr}\left(\mathbf{XY} - \mathbf{XA} - \mathbf{BY}\right)\right) d\mu(\mathbf{X}, \mathbf{Y}) = \exp\left(-\mathrm{Tr}\left(\mathbf{AB}\right)\right). \tag{12}$$

The application of these identities to a moment generating function $g(\nu)$ is known as the *replica trick*, which introduces multiple copies of the Gaussian integration that arises with the expectation operator with respect to $\mathbf{H}_1$ and $\mathbf{H}_2$. We emphasize that the machinery of repeatedly applying the above identities in the evaluation of the expectation requires ν to be a positive integer. We thus will need to assume that our particular $g(\nu)$ can be analytically continued at least in the positive vicinity of zero in the end in order to extract the mean. Note, that this proof is only rigorous in the case that the analytic continuation of $g(\nu)$ to zero is indeed possible. Proving this in turn is a current research topic in mathematics. Nevertheless, all results obtained based on this assumption – including the one derived below – show a perfect match with results obtained through computer simulations.

B. Replica Analysis

According to

$$\mathsf{E}\left[I\right] = \frac{1}{n}\mathsf{E}\left[\ln\det\left(\mathbf{I}_n + \frac{\alpha}{n}\mathbf{H}_2\mathbf{H}_2^{\mathrm{H}} + \frac{\rho\alpha}{n^2}\mathbf{H}_2\mathbf{H}_1\mathbf{H}_1^{\mathrm{H}}\mathbf{H}_2^{\mathrm{H}}\right)\right] - \frac{1}{n}\mathsf{E}\left[\ln\det\left(\mathbf{I}_n + \frac{\alpha}{n}\mathbf{H}_2\mathbf{H}_2^{\mathrm{H}}\right)\right] \triangleq \Psi - \Theta \tag{13}$$

we decompose the evaluation of $\mathsf{E}\left[I\right]$ into two parts. The second term can be seen as the mean mutual information of a point-to-point MIMO channel with an signal-to-noise ratio (SNR) of α. In the large n limit it is well known [7] to evaluate to

$$\Theta = \ln\left(1 + \alpha t\right) + \ln\left(1 + s\right) - st \tag{14}$$

with the unique solution to the equation system

$$s = \frac{\alpha}{1 + \alpha t} \qquad t = \frac{1}{1 + s} \tag{15}$$

yielding positive and real valued coefficients.

It thus remains to evaluate the first term, which is possible via replica analysis. Accordingly, we will firstly evaluate

$$g(\nu) \triangleq \mathsf{E}\left[\det\left(\mathbf{I}_n + \frac{\alpha}{n}\mathbf{H}_2\mathbf{H}_2^{\mathrm{H}} + \frac{\rho\alpha}{n^2}\mathbf{H}_2\mathbf{H}_1\mathbf{H}_1^{\mathrm{H}}\mathbf{H}_2^{\mathrm{H}}\right)^{-\nu}\right] \tag{16}$$

in order to extract the first term in (13) as

$$\Psi = \frac{1}{n}\left(-\frac{\partial}{\partial\nu}\ln g(\nu)\right)\bigg|_{\nu=0}. \tag{17}$$

The above identity is easily verified. Note, that $\ln g(\nu)$ can also be seen as a cumulant generating function.

The expectation operator with respect to the Gaussian matrix $\mathbf{H}_1$ (or analogously $\mathbf{H}_2$) applied to some function f can be written as

$$\mathsf{E}[f(\mathbf{H}_1)] = \int f(\mathbf{H}_1)\exp(-\mathrm{Tr}[\mathbf{H}_1^{\mathrm{H}}\mathbf{H}_1])d\mathbf{H}_1. \tag{18}$$

If f is of the form $\exp(\mathbf{A}^{\mathrm{H}}\mathbf{H}_1 + \mathbf{H}_1^{\mathrm{H}}\mathbf{B})$ the integral can be solved by means of (11) with $\mathbf{M}$ and $\mathbf{N}$ being identity matrices. We arrive at this form by repeatedly applying identity (11) to $g(\nu)$ in the backward direction in the following. If we choose $\mathbf{N} = \mathbf{I}_\nu$ and $\mathbf{M}$ as the argument of the determinant in (16) we can eliminate the determinant by means of an auxiliary matrix $\mathbf{X} \in \mathbb{C}^{n\times\nu}$:

$$g(\nu) = \mathsf{E}\left[\int \exp\left(-\frac{1}{2}\mathrm{Tr}\left[\mathbf{X}^{\mathrm{H}}\mathbf{X} + \frac{\alpha}{n}\mathbf{X}^{\mathrm{H}}\mathbf{H}_2\mathbf{H}_2^{\mathrm{H}}\mathbf{X} + \frac{\rho\alpha}{n^2}\mathbf{X}^{\mathrm{H}}\mathbf{H}_2\mathbf{H}_1\mathbf{H}_1^{\mathrm{H}}\mathbf{H}_2^{\mathrm{H}}\mathbf{X}\right]\right) d\mathbf{X}\right]. \tag{19}$$

In a next step we split the second and third term inside the trace by twice applying (11) – this time with $\mathbf{M}, \mathbf{N}$ identity matrices and $\mathbf{A} = \mathbf{B} = \mathbf{H}_2^{\mathrm{H}}$ in the first application and $\mathbf{A} = \mathbf{B} = \mathbf{H}_1^{\mathrm{H}}\mathbf{H}_2^{\mathrm{H}}$ in the second one, respectively. With $\mathbf{Y}, \mathbf{Z} \in \mathbb{C}^{n\times\nu}$ we thus obtain

$$g(\nu) = \mathsf{E}\Bigg[\int \exp\bigg(-\frac{1}{2}\mathrm{Tr}\Big[\mathbf{X}^{\mathrm{H}}\mathbf{X} + \mathbf{Y}^{\mathrm{H}}\mathbf{Y} + \mathbf{Z}^{\mathrm{H}}\mathbf{Z} + \mathbf{Y}^{\mathrm{H}}\mathbf{H}_1^{\mathrm{H}}\mathbf{H}_2^{\mathrm{H}}\mathbf{X} - \mathbf{X}^{\mathrm{H}}\mathbf{H}_2\mathbf{H}_1\mathbf{Y} + \frac{\alpha}{n}\mathbf{Z}^{\mathrm{H}}\mathbf{H}_2^{\mathrm{H}}\mathbf{X} - \mathbf{X}^{\mathrm{H}}\mathbf{H}_2\mathbf{Z}\Big]\bigg) d\mathbf{X}d\mathbf{Y}d\mathbf{Z}\Bigg]. \tag{20}$$

We repeat this step in order to also split the remaining two products of $\mathbf{H}_1$ and $\mathbf{H}_2$. With $\mathbf{W}_1, \mathbf{W}_2 \in \mathbb{C}^{n\times\nu}$ this yields

$$g(\nu) = \mathsf{E}\Bigg[\int \exp\bigg(-\frac{1}{2}\mathrm{Tr}\Big[\mathbf{X}^{\mathrm{H}}\mathbf{X} + \mathbf{Y}^{\mathrm{H}}\mathbf{Y} + \mathbf{Z}^{\mathrm{H}}\mathbf{Z} + \mathbf{W}_1^{\mathrm{H}}\mathbf{W}_1 + \mathbf{W}_2^{\mathrm{H}}\mathbf{W}_2 + \frac{\rho}{n}\mathbf{Y}^{\mathrm{H}}\mathbf{H}_1^{\mathrm{H}}\mathbf{W}_2 - \frac{\alpha}{n}\mathbf{W}_2^{\mathrm{H}}\mathbf{H}_2^{\mathrm{H}}\mathbf{X} + \mathbf{X}^{\mathrm{H}}\mathbf{H}_2\mathbf{W}_1 + \mathbf{W}_1^{\mathrm{H}}\mathbf{H}_1\mathbf{Y} + \frac{\alpha}{n}\mathbf{Z}^{\mathrm{H}}\mathbf{H}_2^{\mathrm{H}}\mathbf{X} - \mathbf{X}^{\mathrm{H}}\mathbf{H}_2\mathbf{Z}\Big]\bigg) \times d\mathbf{X}d\mathbf{Y}d\mathbf{Z}d\mathbf{W}_1d\mathbf{W}_2\Bigg]. \tag{21}$$

Using the fact that the trace of a product of matrices is invariant under cyclic permutations of the factors we are now ready to perform the integration over $\mathbf{H}_1$ and $\mathbf{H}_2$, i.e., to eliminate the expectation operator by applying identity (11). For the integration over $\mathbf{H}_1$ we choose $\mathbf{A} = \mathbf{W}_1\mathbf{Y}^{\mathrm{H}}$ and $\mathbf{B} = \frac{\rho}{n}\mathbf{W}_2\mathbf{Y}^{\mathrm{H}}$, for the one over $\mathbf{H}_2$ we choose $\mathbf{A} = \mathbf{Z}\mathbf{X}^{\mathrm{H}} + \mathbf{W}_1\mathbf{X}^{\mathrm{H}}$ and $\mathbf{B} = \frac{\alpha}{n}\mathbf{W}_2\mathbf{X}^{\mathrm{H}} + \frac{\alpha}{n}\mathbf{Z}\mathbf{X}^{\mathrm{H}}$ and end up with

$$g(\nu) = \int \exp\bigg(-\frac{1}{2}\mathrm{Tr}\Big[\mathbf{X}^{\mathrm{H}}\mathbf{X} + \mathbf{Y}^{\mathrm{H}}\mathbf{Y} + \mathbf{Z}^{\mathrm{H}}\mathbf{Z} + \mathbf{W}_1^{\mathrm{H}}\mathbf{W}_1 + \mathbf{W}_2^{\mathrm{H}}\mathbf{W}_2 + \frac{1}{2}\Big(-\frac{\rho}{n}\mathbf{Y}^{\mathrm{H}}\mathbf{Y}\mathbf{W}_1^{\mathrm{H}}\mathbf{W}_2 + \frac{\alpha}{n}\mathbf{X}^{\mathrm{H}}\mathbf{X}\mathbf{W}_2^{\mathrm{H}}\mathbf{W}_1 - \frac{\alpha}{n}\mathbf{X}^{\mathrm{H}}\mathbf{X}\mathbf{Z}^{\mathrm{H}}\mathbf{W}_1 - \frac{\alpha}{n}\mathbf{X}^{\mathrm{H}}\mathbf{X}\mathbf{W}_2^{\mathrm{H}}\mathbf{Z} + \frac{\alpha}{n}\mathbf{X}^{\mathrm{H}}\mathbf{X}\mathbf{Z}^{\mathrm{H}}\mathbf{Z}\Big)\Big]\bigg) d\mathbf{W}_1d\mathbf{W}_2d\mathbf{X}d\mathbf{Y}d\mathbf{Z}. \tag{22}$$

Next, we make use of (12) in order to split the quartic terms.

With $\mathbf{R}_1, \mathbf{Q}_1, \ldots, \mathbf{R}_5, \mathbf{Q}_5 \in \mathbb{C}^{\nu\times\nu}$

$$\begin{aligned} g(\nu) = \int \exp\Bigg(-\frac{1}{2}\mathrm{Tr}\Bigg[&\mathbf{X}^{\mathrm{H}}\mathbf{X} + \mathbf{Y}^{\mathrm{H}}\mathbf{Y} + \mathbf{Z}^{\mathrm{H}}\mathbf{Z} \\ &+\mathbf{W}_1^{\mathrm{H}}\mathbf{W}_1 + \mathbf{W}_2^{\mathrm{H}}\mathbf{W}_2 \\ &- (\mathbf{R}_1\mathbf{Q}_1 + \mathbf{R}_2\mathbf{Q}_2 + \mathbf{R}_3\mathbf{Q}_3 + \mathbf{R}_4\mathbf{Q}_4 + \mathbf{R}_5\mathbf{Q}_5) \\ &+\frac{\rho}{n}\mathbf{R}_1\mathbf{Y}^{\mathrm{H}}\mathbf{Y} - \mathbf{Q}_1\mathbf{W}_1^{\mathrm{H}}\mathbf{W}_2 \\ &+\frac{\alpha}{n}\mathbf{R}_2\mathbf{X}^{\mathrm{H}}\mathbf{X} + \mathbf{Q}_2\mathbf{W}_2^{\mathrm{H}}\mathbf{W}_1 \\ &+\frac{\alpha}{n}\mathbf{R}_3\mathbf{X}^{\mathrm{H}}\mathbf{X} - \mathbf{Q}_3\mathbf{Z}^{\mathrm{H}}\mathbf{W}_1 \\ &+\frac{\alpha}{n}\mathbf{R}_4\mathbf{X}^{\mathrm{H}}\mathbf{X} - \mathbf{Q}_4\mathbf{W}_2^{\mathrm{H}}\mathbf{Z} \\ &+\frac{\alpha}{n}\mathbf{R}_5\mathbf{X}^{\mathrm{H}}\mathbf{X} + \mathbf{Q}_5\mathbf{Z}^{\mathrm{H}}\mathbf{Z}\Big)\Bigg]\Bigg) \\ &\times d\mathbf{W}_1 d\mathbf{W}_2 d\mathbf{X} d\mathbf{Y} d\mathbf{Z} d\lambda, \end{aligned} \tag{23}$$

where we have combined the integral measures for the various $\mathbf{R}_k$'s an $\mathbf{Q}_k$'s into the single integral measure

$$d\lambda \triangleq \prod_{k=1}^{5} d\mu(\mathbf{R}_k, \mathbf{Q}_k). \tag{24}$$

We can now get rid of the integrals over $\mathbf{W}_1, \mathbf{W}_2, \mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ by means of (11) again. The integration over $\mathbf{W}_1$ is done precisely as those over $\mathbf{H}_1$ and $\mathbf{H}_2$. With $\mathbf{M} = \mathbf{I}_n$ and $\mathbf{N} = \mathbf{I}_\nu + \frac{\rho}{n}\mathbf{R}_1$ for the integration over $\mathbf{Y}$ and $\mathbf{N} = \mathbf{I}_\nu + \frac{\alpha}{n}(\mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_4 + \mathbf{R}_5)$ for the integration over $\mathbf{X}$ as well as $\mathbf{A} = \mathbf{B} = \mathbf{0}$ in both cases we obtain

$$\begin{aligned} g(\nu) = \int \exp\Bigg(-\frac{1}{2}\mathrm{Tr}\,\big(&\mathbf{Z}^{\mathrm{H}}\mathbf{Z} + \mathbf{W}_2^{\mathrm{H}}\mathbf{W}_2 \\ &- (\mathbf{R}_1\mathbf{Q}_1 + \mathbf{R}_2\mathbf{Q}_2 + \mathbf{R}_3\mathbf{Q}_3 + \mathbf{R}_4\mathbf{Q}_4 + \mathbf{R}_5\mathbf{Q}_5) \\ &+\mathbf{Q}_1\mathbf{Q}_2\mathbf{W}_2^{\mathrm{H}}\mathbf{W}_2 - \mathbf{Q}_1\mathbf{Q}_3\mathbf{Z}^{\mathrm{H}}\mathbf{W}_2 \\ &-\mathbf{Q}_4\mathbf{W}_2^{\mathrm{H}}\mathbf{Z} + \mathbf{Q}_5\mathbf{Z}^{\mathrm{H}}\mathbf{Z} \\ &+\ln\det\left(\mathbf{I}_\nu + \frac{\alpha}{n}(\mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_4 + \mathbf{R}_5)\right) \\ &+\ln\det\left(\mathbf{I}_\nu + \frac{\rho}{n}\mathbf{R}_1\right)\Bigg) d\mathbf{W}_2 d\mathbf{Z} d\lambda. \end{aligned} \tag{25}$$

Similarly, we can also perform the integration over $\mathbf{W}_2$ and $\mathbf{Z}$ (in this order) and essentially arrive at

$$g(\nu) = \int \exp(-\mathcal{S}) \cdot d\lambda, \tag{26}$$

with

$$\begin{aligned} \mathcal{S} = -\mathrm{Tr}\,&(\mathbf{R}_1\mathbf{Q}_1 + \mathbf{R}_2\mathbf{Q}_2 + \mathbf{R}_3\mathbf{Q}_3 + \mathbf{R}_4\mathbf{Q}_4 + \mathbf{R}_5\mathbf{Q}_5) \\ &+n\ln\det\left(\mathbf{I}_\nu + \frac{\rho}{n}\mathbf{R}_1\right) \\ &+n\ln\det\left(\mathbf{I}_\nu + \frac{\alpha}{n}(\mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_4 + \mathbf{R}_5)\right) \\ &+n\ln\det\left(\mathbf{I}_\nu - (\mathbf{I}_\nu + \mathbf{Q}_1\mathbf{Q}_2)^{-1}\mathbf{Q}_4\mathbf{Q}_1\mathbf{Q}_3 + \mathbf{Q}_5\right) \\ &+n\ln\det(\mathbf{I}_\nu + \mathbf{Q}_1\mathbf{Q}_2). \end{aligned} \tag{27}$$

Note that no matrix with one of its dimension equal to n appears in $\mathcal{S}$ anymore.

The last remaining integral in (26) can be solved by means of saddle point integration. We thus need to identify the global minimum of $\mathcal{S}$. The common assumption of *replica symmetry* [14] suggests that all complex matrices need to be proportional to the identity matrix at this point. With

$$\mathbf{R}_1 = r_1 n\mathbf{I}_\nu \quad , \quad \mathbf{Q}_1 = q_1\mathbf{I}_\nu, \tag{28}$$
$$\mathbf{R}_2 = r_2 n\mathbf{I}_\nu \quad , \quad \mathbf{Q}_2 = q_2\mathbf{I}_\nu, \tag{29}$$
$$\mathbf{R}_3 = r_3 n\mathbf{I}_\nu \quad , \quad \mathbf{Q}_3 = q_3\mathbf{I}_\nu, \tag{30}$$
$$\mathbf{R}_4 = r_4 n\mathbf{I}_\nu \quad , \quad \mathbf{Q}_4 = q_4\mathbf{I}_\nu, \tag{31}$$
$$\mathbf{R}_5 = r_5 n\mathbf{I}_\nu \quad , \quad \mathbf{Q}_5 = q_5\mathbf{I}_\nu, \tag{32}$$

at its minimum $\mathcal{S}$ evaluates to

$$\begin{aligned} \mathcal{S}_0 \;=\; \nu\cdot n\cdot\{&\ln(1+\rho r_1) + \ln(1+\alpha(r_2+r_3+r_4+r_5)) \\ &+\ln(1+q_1q_2 - q_1q_3q_4q_2 + q_5 + q_1q_2q_5) \\ &-(r_1q_1 + r_2q_2 + r_3q_3 + r_4q_4 + r_5q_5)\}. \end{aligned} \tag{33}$$

The respective coefficients r_k and q_k have to be chosen such that the expression is indeed minimized. They are found by differentiating (33) for each of them and setting the resulting expressions to zero. The derivatives for the r_k's (note that we can summarize $r_2 + r_3 + r_4 + r_5 \triangleq \tilde{r}_2$ by symmetry) yield

$$0 = q_1 - \frac{\rho}{1+\rho r_1}, \tag{34}$$
$$0 = q_2 - \frac{\alpha}{1+\alpha\tilde{r}_2}, \tag{35}$$
$$0 = q_3 - \frac{\alpha}{1+\alpha\tilde{r}_2}, \tag{36}$$
$$0 = q_4 - \frac{\alpha}{1+\alpha\tilde{r}_2}, \tag{37}$$
$$0 = q_5 - \frac{\alpha}{1+\alpha\tilde{r}_2}. \tag{38}$$

We realize that $q_2 = q_3 = q_4 = q_5$. Taking this into account the derivatives for the q_k's yield

$$0 = r_1 - \frac{q_2}{1+q_1q_2+q_2}, \tag{39}$$
$$0 = \tilde{r}_2 - \frac{1+q_1}{1+q_1q_2+q_2}. \tag{40}$$

(33) thus simplifies to

$$\begin{aligned} \mathcal{S}_0 \;=\; \nu\cdot n\cdot\{&\ln(1+\rho r_1) + \ln(1+\alpha\tilde{r}_2) \\ &+\ln(1+q_2+q_1q_2) - (r_1q_1 + \tilde{r}_2q_2)\} \end{aligned} \tag{41}$$

with

$$q_1 = \frac{\rho}{1+\rho r_1}, \tag{42}$$
$$q_2 = \frac{\alpha}{1+\alpha\tilde{r}_2}, \tag{43}$$
$$r_1 = \frac{q_2}{1+q_1q_2+q_2}, \tag{44}$$
$$\tilde{r}_2 = \frac{1+q_1}{1+q_1q_2+q_2}, \tag{45}$$

where $\mathcal{S}$ is minimized by the (unique) solution for which all coefficients are both real valued and positive.

Since by the saddle point method

$$\lim_{n\to\infty}\frac{1}{n}\ln\int e^{-nf(x_1,\ldots,x_N)}dx_1\cdot\ldots\cdot dx_N = \min_{x_1,\ldots,x_N} f(x_1,\ldots,x_N), \quad (46)$$

for some function f with well defined Hessian at its global minimum, via (17) we finally arrive at

$$\Psi = \ln(1+\rho r_1)+\ln(1+\alpha\tilde{r}_2) + \ln(1+q_2+q_1q_2)-(r_1q_1+\tilde{r}_2q_2), \quad (47)$$

which completes the proof.

V. Comparison with Simulation Results

We verify the results stated in the theorem by means of Monte Carlo simulations. The respective plot is shown in Fig. 1, where we present the ergodic mutual information (not normalized with respect to the spatial dimensions, i.e., $\mathsf{E}[nI]$) versus the SNR for $n = 2, 4$ and 8. We observe that even for only two antennas the approximation is reasonable, for four antennas the match is close to perfect, while for eight antennas no difference between analytic approximation and numeric evaluation can be seen anymore. Our simulation results thus also demonstrate that the replica method – despite its deficiency of not being mathematically rigorous yet – indeed reveals the correct solution to our problem.

VI. Generalizations

The above result is generalized in reference [15]. In particular, we evaluate the cumulant generating function of nI along the procedure formalized in [11], which allows for

- the identification of the residual term of $\mathsf{E}[I]$ as $\mathcal{O}(n^{-2})$,
- the evaluation of the variance of nI in the large n limit,
- and proving asymptotic Gaussianity of nI.

Note that the first point implies that our approximation becomes tight even for $\mathsf{E}[nI]$ as n grows large. We also extend the analysis for correlated fading, different antenna numbers at source, relay and destination and (deterministic) transmit and receive beamforming at the three terminals.

Finally, it should be noted that the above analysis can be extended to channels with more than two hops. The additional sums of products of channel matrices in the signal and signal-plus-noise covariance matrices can be broken down and recomposed analogously to the two-hop case by iteratively applying the above steps.

VII. Conclusion

We have applied the replica method to the two-hop i.i.d. Rayleigh fading amplify-and-forward relay channel without direct link between source and destination. We obtain the asymptotic ergodic mutual information of this channel. Computer experiments show, that the derived expression serves as an excellent approximation even for channels with only very few antennas.

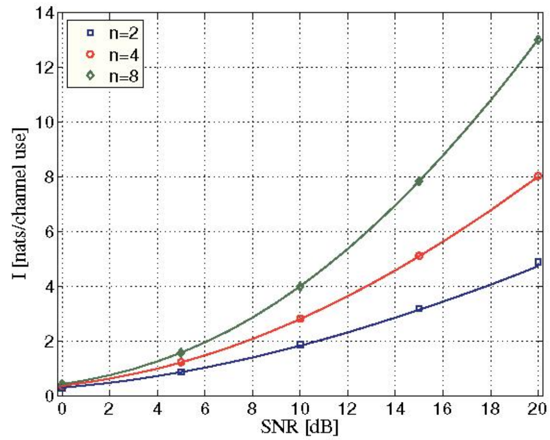

Fig. 1. Ergodic mutual information versus signal-to-noise ratio: the solid lines represent our analytical approximations; circles, squares and diamonds mark the corresponding values as obtained by means of Monte Carlo simulations.

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