Achievable Rate Regions for the Two-way Relay Channel

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***Abstract*— We study the two-way communication problem for the relay channel. Hereby, two terminals communicate simultaneously in both directions with the help of one relay. We consider the *restricted* two-way problem, i.e., the encoders at both terminals do not cooperate. We provide achievable rate regions for different cooperation strategies, such as decode-and-forward based on block Markov superposition coding and compress-and-forward based on Wyner-Ziv source coding. We also evaluate the regions for the special case of additive white Gaussian noise channels. We show that a combined strategy of block Markov superposition coding and Wyner-Ziv coding achieves the cut-set upper bound on the sum-rate of the two-way relay channel when the relay is in the proximity of one of the terminals.**

I. Introduction

In a two-way communication channel two terminals simultaneously transmit their messages to each other and the messages interfere with each other. This channel was first studied by Shannon [1] where he found an inner and an outer bound on the capacity region. It was shown that the inner bound coincides with the capacity region of the *restricted* two-way channel (TWC). In a restricted TWC the encoders of both terminals do not cooperate and the transmitted symbols at one terminal only depend on the message to be transmitted at that terminal and not on the past received symbols. In [2] it was further shown, that the capacity region of the Gaussian TWC coincides with the inner bound, i.e., the encoders do not need to cooperate in order to achieve the capacity region. The capacity region of the general TWC, i.e, when both encoders are allowed to cooperate via their received symbols, is still unknown. In this work we consider the restricted two-way communication problem for the relay channel (TWRC), i.e., two terminals want to exchange messages whereas a third terminal helps both terminals in the communication process. For one-way relay channels, several strategies are known: decode-and-forward [3], [4], compress-and-forward [3], [4] or amplify-and-forward [5]. In this work we are interested in how these strategies can be applied to the TWRC with full-duplex terminals and which rate regions are achievable.

II. System Model

We consider a two-way relay channel (TWRC) with three full-duplex terminals. Terminals $\mathtt{T}_1$ and $\mathtt{T}_2$ want to exchange messages (two-way communication) with the help of terminal $\mathtt{T}_3$ which acts as relay and has no own messages to transmit. The random variables of the channel are:

- the forward message $W \in \mathcal{W}$ of terminal $\mathtt{T}_1$,
- the backward message $V \in \mathcal{V}$ of terminal $\mathtt{T}_2$,
- the channel inputs $X_{ti} \in \mathcal{X}$, $t = 1, 2, 3$, $i = 1, 2, \ldots, N$,
- the channel outputs $Y_{ti} \in \mathcal{Y}$, $t = 1, 2, 3$, $i = 1, 2, \ldots, N$,
- the message estimate $\widehat{W} \in \mathcal{W}$ at terminal $\mathtt{T}_2$,
- the message estimate $\widehat{V} \in \mathcal{V}$ at terminal $\mathtt{T}_1$,

where $\mathcal{W} = \{1, 2, \ldots, 2^{B_W}\}$ and $\mathcal{V} = \{1, 2, \ldots, 2^{B_V}\}$, i.e., message W carries B_W information bits and message V carries B_V information bits. The input alphabet is given by $\mathcal{X}$ and the output alphabet by $\mathcal{Y}$. For the *general* TWRC a transmit symbol of terminal $\mathtt{T}_1$ is a function of its message W and its past channel outputs $Y_1^{i-1} = Y_{11}, Y_{12}, \ldots, Y_{1i-1}$ and a transmit symbol of terminal $\mathtt{T}_2$ is a function of its message V and its past channel outputs $Y_2^{i-1} = Y_{21}, Y_{22}, \ldots, Y_{2i-1}$. In this work we consider the *restricted* TWRC [1] where transmit symbols at terminals $\mathtt{T}_1$ and $\mathtt{T}_2$ are functions of their messages only, i.e., $X_{1i} = f_{1i}(W)$ and $X_{2i} = f_{2i}(V)$ for $i = 1, 2, \ldots, N$. Note that the rate regions obtained for the restricted TWRC serve as inner bounds for the capacity region of the general TWRC. A transmit symbol of relay terminal $\mathtt{T}_3$ is determined by the relay's past channel outputs, i.e., $X_{3i} = f_{3i}(Y_3^{i-1})$ for $i = 1, 2, \ldots, N$. We assume a time invariant and memoryless TWRC which is defined by the conditional channel distribution $p_{Y_1Y_2Y_3|X_1X_2X_3}(y_1, y_2, y_3|x_1, x_2, x_3)$ where X_t and Y_t, $t = 1, 2, 3$ are random variables representing the respective channel inputs and outputs[1]. Terminal $\mathtt{T}_1$ computes its message estimate $\widehat{V}$ as a function of its past channel outputs Y_1^N and its transmitted message W whereas terminal $\mathtt{T}_2$ computes its message estimate $\widehat{W}$ as a function of its past channel outputs Y_2^N and its transmitted message V.

III. Relaying Strategies

In this section we study the rate regions achievable for different relaying strategies.

A. Decode-and-forward

At terminal $\mathtt{T}_1$ the message w is divided into K blocks $w_1, w_2, \ldots, w_K$ of nR_1 bits each. The transmission is performed in $K+1$ blocks, i.e., $B_W = KnR_1$ message bits are transmitted in $N = (K+1)n$ channel uses with an overall

[1]From now on subscripts in the probability distributions are dropped when they are obvious by inspection of the arguments.

rate $R_1K/(K+1)$. The codeword of length n at terminal $\mathtt{T}_1$ consists of a superposition of two codewords:

$$x_1^n(i,j) = \sqrt{\alpha_1}x_{3\mathrm{f}}^n(i) + \sqrt{\beta_1}\widetilde{x}_1^n(j) \tag{1}$$

where i and j range from 1 to 2^{nR_1} and α_1, β_1 are scaling coefficients. The $n \cdot 2^{nR_1}$ symbols $\widetilde{x}_{1m}(j)$, $m = 1 \ldots, n$, are chosen independently according to $P_{\widetilde{X}_1}(\cdot)$ and the $n \cdot 2^{nR_1}$ symbols $x_{3\mathrm{f}m}(i)$ are chosen independently according to $P_{X_{3\mathrm{f}}}(\cdot)$. The same procedure is performed at terminal $\mathtt{T}_2$: the message v is divided into K blocks $v_1, v_2, \ldots, v_K$ of nR_2 bits each and transmitted in $K+1$ blocks. The codeword at terminal $\mathtt{T}_2$ consists of two codewords:

$$x_2^n(p,q) = \sqrt{\alpha_2}x_{3\mathrm{b}}^n(p) + \sqrt{\beta_2}\widetilde{x}_2^n(q) \tag{2}$$

where p and q range from 1 to 2^{nR_2} and α_2, β_2 are scaling coefficients. The $n \cdot 2^{nR_2}$ symbols $\widetilde{x}_{2m}(q)$ are chosen independently according to $P_{\widetilde{X}_2}(\cdot)$ and the $n \cdot 2^{nR_2}$ symbols $x_{3\mathrm{b}m}(p)$ are chosen independently according to $P_{X_{3\mathrm{b}}}(\cdot)$. Note that for $K \to \infty$ and for fixed n the rates $R_1K/(K+1)$ and $R_2K/(K+1)$ are arbitrary close to R_1 and R_2, respectively.

Block Markov Superposition Coding. In the first block, $k=1$, terminals $\mathtt{T}_1$ and $\mathtt{T}_2$ transmit $x_1^n(1,w_1)$ and $x_2^n(1,v_1)$, respectively. The relay terminal $\mathtt{T}_3$ transmits $x_3^n(1,1) = x_{3\mathrm{f}}^n(1) + x_{3\mathrm{b}}^n(1)$, where $x_{3\mathrm{f}}^n(1)$ denotes the relay's codeword associated to the *forward* direction $\mathtt{T}_1 \to \mathtt{T}_2$ and $x_{3\mathrm{b}}^n(1)$ the codeword associated to the *backward* direction $\mathtt{T}_1 \leftarrow \mathtt{T}_2$. Terminal $\mathtt{T}_3$ is able to decode w_1 and v_1 as long as

$$R_1 \leq I(X_1;Y_3|X_2X_3) \tag{3}$$
$$R_2 \leq I(X_2;Y_3|X_1X_3) \tag{4}$$
$$R_1 + R_2 \leq I(X_1X_2;Y_3|X_3) \tag{5}$$

and the block size n is large. In the second block, $k=2$, terminals $\mathtt{T}_1$ and $\mathtt{T}_2$ transmit $x_1^n(w_1,w_2)$ and $x_2^n(v_1,v_2)$, respectively. The relay terminal transmits $x_3^n(w_1,v_1) = x_{3\mathrm{f}}^n(w_1) + x_{3\mathrm{b}}^n(v_1)$. Terminal $\mathtt{T}_3$ is able to decode w_2 and v_2 as long as n is large and (3)–(5) are true. One continues in this way until block $K+1$, where terminals $\mathtt{T}_1$ and $\mathtt{T}_2$ transmit $x_1^n(w_K,1)$ and $x_2^n(v_K,1)$, respectively, and where terminal $\mathtt{T}_3$ transmits $x_3^n(w_K,v_K) = x_{3\mathrm{f}}^n(w_K) + x_{3\mathrm{b}}^n(v_K)$.

Backward Decoding. Let $Y_1^n(k)$ and $Y_2^n(k)$ be the observed symbols at terminal $\mathtt{T}_1$ and $\mathtt{T}_2$ in block k, respectively. Starting in the last block, $k=K+1$, terminal $\mathtt{T}_2$ knows v_K (*back-propagating self-interference*) and decodes the message w_K from $Y_2^n(K+1)$ as long as

$$R_1 \leq I(X_1X_3;Y_2|X_2) \tag{6}$$

and the block size n is large. Similarly, terminal $\mathtt{T}_1$ knows w_K and decodes v_K from $Y_1^n(K+1)$ as long as

$$R_2 \leq I(X_2X_3;Y_1|X_1). \tag{7}$$

In block $k=K$, terminal $\mathtt{T}_2$ knows w_K and v_{K-1} and decodes w_{K-1} from $Y_2^n(K)$ as long as (6) is true and n sufficiently large. Similar for terminal $\mathtt{T}_1$. One continues this way until block $k=1$ where all messages $w_1, \ldots, w_K$ and $v_1, \ldots, v_K$ are decoded. Combining (3)–(7) we obtain the achievable rate region for this scheme in the following proposition.

Proposition 1 (Two-way decode-and-forward) An achievable rate region of the TWRC is given by $\bigcup \{R_1, R_2\}$ with

$$R_1 \leq \min\left(I(X_1;Y_3|X_2X_3), I(X_1X_3;Y_2|X_2)\right) \tag{8}$$
$$R_2 \leq \min\left(I(X_2;Y_3|X_1X_3), I(X_2X_3;Y_1|X_1)\right) \tag{9}$$
$$R_1 + R_2 \leq I(X_1X_2;Y_3|X_3) \tag{10}$$

where the union is over all product distributions $p(x_1|x_{3\mathrm{f}})p(x_2|x_{3\mathrm{b}})p(x_{3\mathrm{f}})p(x_{3\mathrm{b}})$.

Next, we establish the rate region of the memoryless additive white Gaussian noise TWRC. The received symbols at terminals $\mathtt{T}_1$, $\mathtt{T}_2$ and $\mathtt{T}_3$ are defined by

$$Y_1 = h_0X_2 + h_1X_3 + Z_1 \tag{11}$$
$$Y_2 = h_0X_1 + h_2X_3 + Z_2 \tag{12}$$
$$Y_3 = h_1X_1 + h_2X_2 + Z_3 \tag{13}$$

where $h_0, h_1, h_2 \in \mathbb{R}$ denote deterministic channel gains[2] between terminals $\mathtt{T}_1$ and $\mathtt{T}_2$, terminals $\mathtt{T}_1$ and $\mathtt{T}_3$, and terminals $\mathtt{T}_2$ and $\mathtt{T}_3$, respectively. The $Z_i \in \mathbb{R}$ denote independent Gaussian random variables with zero mean and unit variances. All input and output symbols are taken from the real alphabet $\mathbb{R}$. We impose the per-symbol power constraints $\mathbb{E}[X_{3\mathrm{f}}^2] \leq \gamma P_3$, $\mathbb{E}[X_{3\mathrm{b}}^2] \leq \overline{\gamma} P_3$ and $\mathbb{E}[\widetilde{X}_i^2] \leq P_i$, where $i = 1,2$ and $\overline{\gamma} = (1-\gamma)$. Let ρ_1 be the correlation coefficient between codeword x_1^n and x_3^n and ρ_2 the correlation coefficient between

[2]We assume reciprocity for all channel gains

Proposition 2 (Two-way decode-and-forward, AWGN)

$$\bigcup_{\substack{0\leq\rho_1,\rho_2\leq 1\\ 0<\gamma<1}} \Bigg\{ R_1, R_2 : \; R_1 \leq \min\left(C\left(P_1h_1^2\left(1-\rho_1^2\right)\right), C\left(P_1h_0^2 + \gamma P_3h_2^2 + 2\rho_1\sqrt{\gamma P_1P_3}h_0h_2\right)\right) \tag{14}$$

$$R_2 \leq \min\left(C\left(P_2h_2^2\left(1-\rho_2^2\right)\right), C\left(P_2h_0^2 + \overline{\gamma} P_3h_1^2 + 2\rho_2\sqrt{\overline{\gamma} P_2P_3}h_0h_1\right)\right) \tag{15}$$

$$R_1 + R_2 \leq C\left(P_1h_1^2\left(1-{\rho_1}^2\right) + P_2h_2^2\left(1-{\rho_2}^2\right)\right) \Bigg\} \tag{16}$$

codeword x_2^n and x_3^n, defined as $\rho_1 = \mathbb{E}[X_1X_3]/\sqrt{P_1P_3}$ and $\rho_2 = \mathbb{E}[X_2X_3]/\sqrt{P_2P_3}$. The scaling coefficients α_1 and β_1 for the codeword of terminal $\mathtt{T}_1$, see (1), are chosen as $\alpha_1 = \rho_1^2P_1/(\gamma P_3)$ and $\beta_1 = 1-\rho_1^2$. The scaling coefficients α_2 and β_2 for the codeword of terminal $\mathtt{T}_2$, see (2), are correspondingly chosen as $\alpha_2 = \rho_2^2P_2/(\overline{\gamma}P_3)$ and $\beta_2 = 1-\rho_2^2$. These scaling coefficients ensure that the average power of codeword x_1^n is P_1 and of x_2^n is P_2. The average power of codeword x_3^n is P_3, whereas γP_3 is used for the *forward* codeword $x_{3\mathrm{f}}^n$ and $\overline{\gamma}P_3$ for the *backward* codeword $x_{3\mathrm{b}}^n$. The rate region for the Gaussian TWRC is then given in Proposition 2 with $C(x) = \frac{1}{2}\log_2(1+x)$.

When γ is zero, the relay is not used for the forward direction, but terminal $\mathtt{T}_1$ can still communicate with terminal $\mathtt{T}_2$ over the direct link with the rate being $C(P_1h_0^2)$. Similar for $\gamma = 1$, the relay is then used only for the forward direction, but not for the backward direction. The rate for the backward direction is then given by $C(P_2h_0^2)$.

B. Compress-and-forward

In the previous section the relay was enforced to decode the symbols from both terminals. Another scheme where the relay does not decode but sends estimates of its received symbols was developed in [3, Theorem 6] for the one-way relay channel and is nowadays known as *compress-and-forward*, see also [4], [6]. We apply this scheme to the TWRC to obtain the following rate region.

Proposition 3 (Two-way compress-and-forward) An achievable rate region of the TWRC is given by $\bigcup\left\{R_1, R_2\right\}$ with

$$R_1 \le I(X_1; Y_2\widehat{Y}_3|X_2X_3) \tag{18}$$
$$R_2 \le I(X_2; Y_1\widehat{Y}_3|X_1X_3) \tag{19}$$

subject to the constraint

$$\begin{aligned}\max\left(I(\widehat{Y}_3; Y_3|X_1X_3Y_1), I(\widehat{Y}_3; Y_3|X_2X_3Y_2)\right)\\ < \min\left(I(X_3; Y_1|X_1), I(X_3; Y_2|X_2)\right)\end{aligned} \tag{20}$$

where the union is over all distributions

$$p(x_1)p(x_2)p(x_3)p(y_1,y_2,y_3|x_1,x_2,x_3)p(\widehat{y}_3|x_3,y_3). \tag{21}$$

An outline of the proof is given in the Appendix. The auxiliary random variable $\widehat{Y}_3$ represents a quantized and compressed version of Y_3 that is available at terminals $\mathtt{T}_1$ and $\mathtt{T}_2$ since the constraint in (20) ensures that the index of the quantized codeword $\widehat{Y}_3^n$ is transmitted reliably to *both* terminals $\mathtt{T}_1$ and $\mathtt{T}_2$. Next, we evaluate the rate region given in (18)–(20) to the special case of additive Gaussian noise channels defined in (11)–(13).

Proposition 4 (Two-way compress-and-forward, AWGN) An achievable rate region of the Gaussian TWRC is given by $\bigcup\left\{R_1, R_2\right\}$ with

$$R_1 \le C\left(1 + P_1h_0^2 + \frac{P_1h_1^2}{1+\sigma_\mathrm{c}^2}\right) \tag{22}$$
$$R_2 \le C\left(1 + P_2h_0^2 + \frac{P_2h_2^2}{1+\sigma_\mathrm{c}^2}\right) \tag{23}$$

subject to the constraint

$$R_3 \le \min\left(C\left(\frac{P_3h_2^2}{1+P_1h_0^2}\right), C\left(\frac{P_3h_1^2}{1+P_2h_0^2}\right)\right). \tag{24}$$

The union is over all variances of the compression noise that satisfies $\sigma_\mathrm{c}^2 \ge \max\left\{\sigma_{\mathrm{c}1}^2, \sigma_{\mathrm{c}2}^2\right\}$ with

$$\sigma_{\mathrm{c}1}^2 = \frac{\left(1+P_2h_0^2\right)\left(1+P_2h_2^2\right) - (P_2h_0h_2)^2}{(2^{2R_3}-1)\left(1+P_2h_0^2\right)} \tag{25}$$
$$\sigma_{\mathrm{c}2}^2 = \frac{\left(1+P_1h_0^2\right)\left(1+P_1h_1^2\right) - (P_1h_0h_1)^2}{(2^{2R_3}-1)\left(1+P_1h_0^2\right)}. \tag{26}$$

C. Decode/Compress-and-forward

In [4], [7] and [8] it was shown that for one-way relay channels decode-and-forward achieves the cut-set bound [9] when the source-relay channel is strong. When the relay-destination channel is strong, then compress-and-forward achieves the cut-set bound. In the two-way relay channel we have the situation that when the relay is in the proximity of one of the relays (see Fig.1), employing a pure decode-and-forward strategy or a pure compress-and-forward strategy as given in Propositions 2 and 4 results in a low rate for one of the communication directions. For example, when the relay terminal is in the proximity of terminal $\mathtt{T}_1$, using a decode-and-forward strategy gives a large R_1 but a small R_2. Using a compress-and-forward strategy gives a large value for R_2 but a small value for R_1. A better strategy is to combine both methods: if the relay is near to terminal $\mathtt{T}_1$ it decodes first the message from terminal $\mathtt{T}_1$, treating the signal from terminal $\mathtt{T}_2$ as interference. It subtracts the influence of the decoded message on its received signal, then quantizes and compresses the remaining signal. When the relay is near to terminal $\mathtt{T}_2$ it is the other way around, i.e., compress-and-forward in the forward direction and decode-and-forward in the backward direction. The achievable rate region for this strategy is given in the next proposition directly for Gaussian noise channels.

Proposition 5 (Two-way decode/compress-and-forward) For $h_1 \ge h_2$ we have $\bigcup_{0\le\rho_1\le 1, 0<\gamma<1}\left\{R_1, R_2\right\}$ with

$$R_1 \le R_{1,\mathrm{DF}} \tag{27}$$
$$R_2 \le R_{2,\mathrm{CF}} \tag{28}$$

where $\sigma_\mathrm{c}^2 \ge \frac{1+P_2h_0^2+P_2h_2^2}{(1-\gamma)P_3h_1^2}$ and $R_{1,\mathrm{DF}}$ is given in (31) and $R_{2,\mathrm{CF}}$ in (23). For $h_1 < h_2$ we have $\bigcup_{0\le\rho_2\le 1, 0<\gamma<1}\left\{R_1, R_2\right\}$ with

$$R_1 \le R_{1,\mathrm{CF}} \tag{29}$$
$$R_2 \le R_{2,\mathrm{DF}} \tag{30}$$

where $\sigma_\mathrm{c}^2 \ge \frac{1+P_1h_0^2+P_1h_1^2}{\gamma P_3h_2^2}$ and $R_{2,\mathrm{DF}}$ given in (32) and $R_{1,\mathrm{CF}}$ in (22).

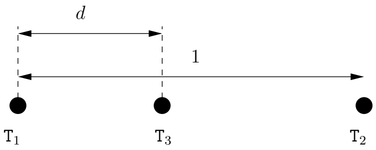

Fig. 1. One-dimensional linear relay network

D. Amplify-and-forward

A popular and simple strategy is to let the relay forward an amplified version of its receive signal. The receive signal at symbol time i at the relay is given as $Y_3[i] = h_1 X_1[i] + h_2 X_2[i] + Z_3[i]$. The receive signal at terminal $\mathtt{T}_2$ is then

$$\begin{aligned} Y_2[i] &= h_0 X_1[i] + g h_2 Y_3[i-1] + Z_2[i] \\ &= h_0 X_1[i] + g h_1 h_2 X_1[i-1] + Z_2[i] + g h_2 Z_3[i-1] \end{aligned} \quad (33)$$

assuming that terminal $\mathtt{T}_2$ can subtract the contribution of its own transmitted signal $X_2[i-1]$. The receive signal at terminal $\mathtt{T}_1$ follows correspondingly as

$$Y_1[i] = h_0 X_2[i] + g h_1 h_2 X_2[i-1] + Z_1[i] + g h_1 Z_3[i-1]. \quad (34)$$

The relay gain is chosen such that the average transmit power of the relay terminal is P_3, i.e., $g(\alpha) = \sqrt{P_3/(\alpha P h_1^2 + (1-\alpha) P h_2^2 + 1)}$ where $P_1 = \alpha P$, $P_2 = (1-\alpha)P$ and $0 \leq \alpha \leq 1$. From (33) and (34) we see that the two-way relay channel with an AF relay decouples into two parallel channels, each being a one-tap inter-symbol interference channel.

Proposition 6 (Two-way amplify-and-forward) An achievable rate region of the Gaussian TWRC is given by the convex hull of $\bigcup_{0 \leq \alpha \leq 1} (R_1, R_2)$ satisfying

$$R_1 \leq C\left(\frac{c_1 a + \sqrt{(1 + c_1 a)^2 - c_1^2 b^2}}{2}\right) \quad (35)$$

$$R_2 \leq C\left(\frac{c_2 a + \sqrt{(1 + c_2 a)^2 - c_2^2 b^2}}{2}\right) \quad (36)$$

where $a = h_0^2 + g(\alpha)^2 h_1^2 h_2^2$, $b = 2 g(\alpha) h_1 h_2$, $c_1 = \frac{P_1}{(1+g(\alpha)^2 h_2^2)}$ and $c_2 = \frac{P_2}{(1+g(\alpha)^2 h_1^2)}$. Furthermore, the rate region defined by (35) and (36) coincides with the capacity region of the nonrestricted Gaussian TWRC with an amplify-and-forward relay.

The rates (35) and (36) are obtained by applying the standard capacity formula for frequency-selective channels [9]. The proof that the rate region is the capacity region of the nonrestricted Gaussian amplify-and-forward TWRC is similar to [2]. Note that the single-user rates are achieved for $\alpha = 0$ and $\alpha = 1$.

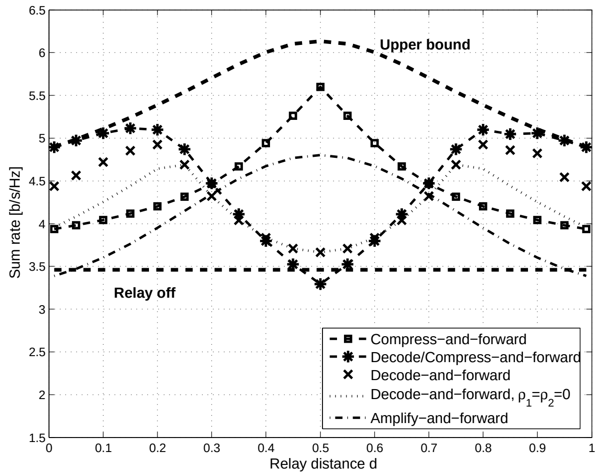

Fig. 2. Sum-rates for different cooperation strategies

IV. Numerical Examples

We compare the different strategies with respect to sum-rate for the simple network geometry depicted in Fig.1. We assume the channel gains to be $h_0 = 1$, $h_1 = 1/d^{\alpha/2}$, $h_2 = 1/(1-d)^{\alpha/2}$, where d is the distance between terminal $\mathtt{T}_1$ and relay terminal $\mathtt{T}_3$ and $\alpha = 3$ is the path loss exponent. We choose $P_1 = P_2 = P_3 = 10$ and zero mean, unit variance Gaussian noise variables. In Fig.2 we compare the sum-rate of the different strategies with the cut-set upper bound of the TWRC[3]. We see that the combined strategy of Proposition 5 achieves the cut-set bound, when the relay is in the proximity of terminal $\mathtt{T}_1$ or $\mathtt{T}_2$ but has loose performance when the relay is somewhere in the middle. In this region the best performance is achieved by the compress-and-forward strategy of Proposition 4. We see that the amplify-and-forward strategy also performs well in this region. The reason is, that in both strategies the relay has not to decode the messages from terminal $\mathtt{T}_1$ and $\mathtt{T}_2$ and is therefore not limited by the multiple access sum-rate. The decode-and-forward strategy based on superposition coding performs better than ordinary decode-and-forward (without coherent cooperation of relay and terminals) only, when the relay is not too far away from one of the terminals. In the middle region both schemes coincide because the optimal correlation coefficients become zero. For comparison, we also plotted the sum-rate, when the relay does not help in the communication process. We see that in most cases the cooperative strategies outperform the noncooperative strategy. For all curves the parameters ρ_1, ρ_2, γ

[3]Application of [9, Theorem 14.10.1] to the TWRC is straightforward

$$R_{1,\mathrm{DF}} = \min\left(C\left(\frac{P_1 h_1^2 \left(1-\rho_1^2\right)}{1 + P_2 h_2^2}\right), C\left(P_1 h_0^2 + \gamma P_3 h_2^2 + 2\rho_1 \sqrt{\gamma P_1 P_3} h_0 h_2\right)\right) \quad (31)$$

$$R_{2,\mathrm{DF}} = \min\left(C\left(\frac{P_2 h_2^2 \left(1-\rho_2^2\right)}{1 + P_1 h_1^2}\right), C\left(P_2 h_0^2 + \overline{\gamma} P_3 h_1^2 + 2\rho_2 \sqrt{\overline{\gamma} P_2 P_3} h_0 h_1\right)\right) \quad (32)$$

have been optimized numerically.

V. Conclusions

We studied achievable rate regions for the restricted two-way relay channel. We have seen that the best strategy in terms of sum-rate is to use a combined decode/compress-and-forward scheme, when the relay terminal is near to one of the terminals. For the case that the channel gains from the relay to both terminals are similar in their amplitude, its better to use a compress-and-forward or amplify-and-forward strategy.

Appendix
Proof of Proposition 3

We provide an outline of the proof which follows [3] and [6] but is modified such that it applies to the TWRC.

Random Codebooks. Terminal $\mathtt{T}_1$ chooses 2^{nR_1} i.i.d. x_1^n each with probability $p(x_1^n) = \prod_{i=1}^n p(x_{1i})$ and labels the codewords by $w \in [1, 2^{nR_1}]$. Terminal $\mathtt{T}_2$ chooses 2^{nR_2} i.i.d. x_2^n each with probability $p(x_2^n) = \prod_{i=1}^n p(x_{2i})$ and labels the codewords by $v \in [1, 2^{nR_2}]$. The relay terminal $\mathtt{T}_3$ chooses 2^{nR_3} i.i.d. x_3^n each with probability $p(x_3^n) = \prod_{i=1}^n p(x_{3i})$ and labels the codewords by $s \in [1, 2^{nR_3}]$. The relay chooses then, for each $x_3^n(s)$, $2^{n(R_3+R_3^{'})}$ i.i.d. $\widehat{Y}_3^n$ each with probability $p(\widehat{y}_3^n|x_3^n(s)) = \prod_{i=1}^n p(\widehat{y}_{3i}|x_{3i}(s))$ and labels the $\widehat{y}_3^n(q, r|s)$ by $q \in [1, 2^{nR_3^{'}}]$ (bin index) and $r \in [1, 2^{nR_3}]$ (relative index within bin). The auxiliary random variable $\widehat{Y}_3$ represents a quantized version of Y_3 and the joint distribution of the random variables $X_1, X_2, X_3, Y_1, Y_2, Y_3, \widehat{Y}_3$ factors as $p(x_1)p(x_2)p(x_3)p(y_1, y_2, y_3|x_1, x_2, x_3)p(\widehat{y}_3|x_3, y_3)$. Note that the channel inputs X_1 and X_2 are independent because we consider the restricted two-way relay channel. In general, these random variables may be dependent [1].

Encoding. At terminal $\mathtt{T}_1$ the message w of nR_1K bits is divided into K equally-sized blocks $w_1, w_2, \ldots, w_K$ of nR_1 bits each. In block k, $k = 1, 2, \ldots, K+1$ terminal $\mathtt{T}_1$ transmits $x_1^n(w_k)$ where $w_{K+1} = 1$. At terminal $\mathtt{T}_2$ the message v of nR_2K bits is divided into K equally-sized blocks $v_1, v_2, \ldots, v_K$ of nR_2 bits each. In block k, $k = 1, 2, \ldots, K+1$ terminal $\mathtt{T}_2$ transmits $x_2^n(v_k)$ where $v_{K+1} = 1$. After block k the relay terminal $\mathtt{T}_3$ observes y_{3k}^n and tries to find a (q_k, r_k) such that $(\widehat{y}_3^n(q_k, r_k|s_k), y_{3k}^n, x_3^n(s_k))$ are jointly typical, sets $s_{k+1} = r_k$ and transmits $x_3(s_{k+1})$ (the codeword that represents the bin index). Notice that in block $k = 1$ the relay sets $s_1 = 1$. The relay is able to find a (q_k, r_k) with high probability when $R_3 + R_3^{'} > I(\widehat{Y}_3; Y_3|X_3)$ and the block size n is large. Roughly spoken, the relation says, that the relay should not compress y_3^n too much.

Decoding. After block k terminal $\mathtt{T}_1$ tries to estimate the bin index s_k by looking for the $x_3^n(s_k)$ that is typical with y_{1k}^n. Terminal $\mathtt{T}_2$ does the same, i.e., looks for the $x_3^n(s_k)$ that is typical with y_{2k}^n. Both terminals succeed with high probability if

$$R_3 < \min(I(X_3; Y_1|X_1), I(X_3; Y_2|X_2)) \tag{37}$$

and the block size n is large. The relation says, that the quantized relay observation (bin index $s_k = q_{k-1}$) has to be communicated reliably to *both* terminals $\mathtt{T}_1$ and $\mathtt{T}_2$. Then terminal $\mathtt{T}_1$ uses $y_{1(k-1)}^n$ to find a r_{k-1} such that $(\widehat{y}_3^n(q_{k-1}, r_{k-1}|s_{k-1}), y_{1(k-1)}^n, x_3^n(s_{k-1})|x_2^n(v_{k-2}))$ is jointly typical. Terminal $\mathtt{T}_2$ does the same, i.e., uses $y_{2(k-1)}^n$ to find r_{k-1} such that $(\widehat{y}_3^n(q_{k-1}, r_{k-1}|s_{k-1}), y_{2(k-1)}^n, x_3^n(s_{k-1})|x_1^n(w_{k-2}))$ is jointly typical. Both terminals succeed with high probability when

$$R_3^{'} < \min(I(\widehat{Y}_3; Y_1|X_1X_3), I(\widehat{Y}_3; Y_2|X_2X_3)) \tag{38}$$

and the block size n is large. This means that there shouldn't be too much codewords in one bin, otherwise the decoders at terminals $\mathtt{T}_1$ and $\mathtt{T}_2$ are not able to resolve the uncertainty with help of their side information Y_1 and Y_2, respectively. Since the quantization has to work for both terminals, the number $2^{nR_3^{'}}$ of codewords per bin is chosen such, that the weaker terminal may decode the quantized codeword $\widehat{Y}_3^n$. Finally, terminal $\mathtt{T}_1$ uses both $y_{1(k-1)}^n$ and $\widehat{y}_3^n$ to find an index $\widehat{v}_{k-1}$ such that $(x_2^n(v_{k-1}), \widehat{y}_3(q_{k-1}, r_{k-1}|s_{k-1}), y_{1(k-1)}^n, x_3^n(s_{k-1}))$ is jointly typical. Again, terminal $\mathtt{T}_2$ does the same, i.e., uses both $y_{2(k-1)}^n$ and $\widehat{y}_3^n$ to find an index $\widehat{w}_{k-1}$ such that $(x_1^n(w_{k-1}), \widehat{y}_3(q_{k-1}, r_{k-1}|r_{k-1}), y_{2(k-1)}^n, x_3^n(s_{k-1}))$ is jointly typical. Both terminals succeed with high probability when

$$R_1 < I(X_1; Y_2\widehat{Y}_3|X_2X_3) \tag{39}$$

$$R_2 < I(X_2; Y_1\widehat{Y}_3|X_1X_3) \tag{40}$$

and the block size n is large. By choosing $R_3^{'} = \min\{I(\widehat{Y}_3; Y_1|X_1X_3), I(\widehat{Y}_3; Y_2|X_2X_3)\} - \epsilon$ we get

$$\begin{aligned} R_3 &> I(\widehat{Y}_3; Y_3|X_3) \\ &\quad - \min\Big(I(\widehat{Y}_3; Y_1|X_1X_3), I(\widehat{Y}_3; Y_2|X_2X_3)\Big) + \epsilon \quad (41) \\ &= \max\Big(I(\widehat{Y}_3; Y_3|X_1X_3Y_1), I(\widehat{Y}_3; Y_3|X_2X_3Y_2)\Big) \quad (42) \end{aligned}$$

The rates of Proposition 3 are then given in (39) and (40), and the constraint (20) follows by combining (37) and (42).

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