Multihop-Enabled Orthogonalization in Distributed MIMO Networks

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***Abstract*—It has recently been proposed to orthogonalize the data streams of n interfering source-destination pairs by employing an intermediate stage of non-cooperating coherent amplify-and-forward relays. This distributed zero-forcing scheme achieves a spatial multiplexing gain $n/2$ and does not require multi-antenna nodes. In this paper we generalize the concept from a single to multiple relay stages. Due to the concatenation of relay stages, the gain allocation problem becomes nonlinear. We show that the zero-forcing relay gain coefficients are given by the common roots of a set of polynomials, and we identify the set of network configurations for which such roots exist. In general, there exist multiple common roots. Based on experimental results we study the impact of the root selection and the number of relay stages on achievable rates. Finally, we propose a new scheme to achieve spatial orthogonalization in a single-hop interference network. Our proposal is based on the observation, that the mathematical model of the multihop problem is identical with the one of a single-hop network if we allow signals to be bounced forth and back between sources and destinations.**

I. Introduction

Cooperative relaying schemes are considered as promising candidates for next generation wireless systems, e.g. upcoming cellular and ad-hoc networks. The density of wireless nodes in such networks is expected to increase significantly in the future, thus rendering node cooperation a key enabler for enhancing data rates, link reliability or coverage of these systems. Extension of coverage is expected to be one of the main upcoming challenges, in particular when systems are operated at higher carrier frequencies (> 5 GHz). One motivation for resorting to such carrier frequencies is their potential of enabling the integration of large antenna arrays even in mobile stations, thus providing additional multiple-input-multiple-output (MIMO) gains. However, as a consequence in cellular networks, cell sizes must be reduced significantly compared to current 2G and 3G networks [1], [2]. Using relays and multihop strategies for assisting transmission in cellular networks has been shown to be an effective means for coverage improvement [3]. This means that higher data rates can be carried over larger distances, thus reducing the required base station density in a cellular network.

In this contribution, we investigate a scenario where several stages of relay nodes assist the communication between a cluster of n source nodes and a cluster of n destination nodes making up n source-destination pairs (cf. Fig. 1). The source signals reach the destination nodes by successively traversing all relay stages in the network on a common physical channel (frequency flat and slow fading), and thus are subject to inter-stream interference in general. Reference [4] presented a method for canceling the inter-stream interference in two-hop networks in a completely decentralized fashion by employing an amplify-and-forward architecture with a certain relay gain allocation. As a generalization of this method, we aim to come up with amplify-and-forward strategies ensuring inter-stream interference cancellation in multihop networks of arbitrary length.

A main result of this paper is the insight that the total number of relay nodes required for the orthogonalization of the source-destination pairs is hardly affected by the number of hops separating source and destination cluster. This implies that the considerable constraint on the required number of relays in the two-hop network can be relaxed significantly, in the sense that the relay nodes can be accumulated over multiple hops in the network. This, in turn, results into a smaller number of relays required per stage. In particular, we shall see that n relay nodes per stage suffice to provide network orthogonalization in a sufficiently long network.

In a second step, we apply our findings on multihop networks to multi-stream interference cancellation in single-hop interference networks. Our approach is to mimic a multihop network by transmitting signals forth and back between sources and destinations several times, thus making the respective interference cancellation techniques applicable. This method is shown to provide a spatial multiplexing gain linear in n if source and/or destination cluster are amended by additional relay nodes. In a cellular environment, e.g., this method could be employed for inter-cell interference cancellation in both up- and down-link.

Conceptually all results obtained for single antenna nodes in this contribution carry over to multi-antenna nodes in a straightforward fashion.

The paper is structured as follows: Section II formally introduces the multihop setting and discusses under which conditions distributed zero-forcing of the source-destination pairs is feasible. In Section III we transfer the distributed multihop zero-forcing concept to single-hop interference networks. Section IV provides an experimental performance analysis. We provide some concluding remarks in Section V.

II. Multihop Relay Networks

A. I-O-Relation and Zero-Forcing Constraints

We consider a network consisting of $L + 2$ clusters: a source cluster $\mathcal{S} = \{\mathrm{S}_1, \ldots, \mathrm{S}_n\}$ and a destination cluster

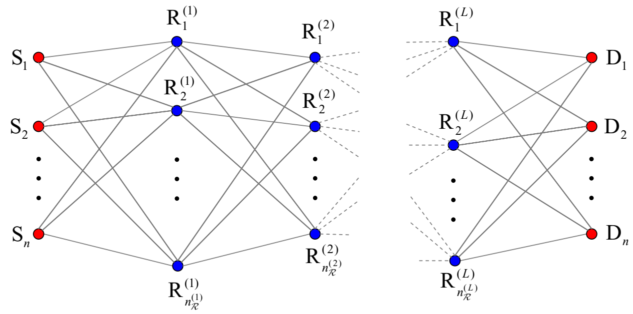

Fig. 1. Network Graph.

$\mathcal{D} = \{\mathrm{D}_1, \dots, \mathrm{D}_n\}$ containing n nodes each, as well as L relay stages $\mathcal{R}_l = \{\mathrm{R}_1^{(l)}, \dots, \mathrm{R}^{(l)}_{n_\mathcal{R}^{(l)}}\}$, $l = 1, \dots, L$, containing $n_\mathcal{R}^{(l)}$ nodes in stage l. All nodes are equipped with a single antenna and nodes within the same cluster are assumed not to exchange any information about their receive signals. We consider transmission of a single codeword per source-destination pair. Transmission is divided into $L+1$ time slots and initiated by the source cluster in the first time slot. In time slot l, relay stage $\mathcal{R}_l$ receives signals from relay stage $\mathcal{R}_{l-1}$ (or the source cluster $\mathcal{S}$ if $l = 1$). Each relay $\mathrm{R}_k^{(l)} \in \mathcal{R}_l$ scales and phase rotates its receive signal, i.e. performs a multiplication with a complex scalar $g_{\mathrm{R}_k^{(l)}}$ before re-transmission in time slot $l+1$. In time slot $L+1$ the destination cluster receives the transmission of $\mathcal{R}_L$. After at most $L+1$ time slots the source cluster can inject new signals into the network without posing interference to the transmission of the previous codeword. The considered setting is depicted in Fig. 1.

We assume a slow and frequency flat fading channel model and denote by h_{IJ} the multiplicative fading coefficient describing transmission from node J to node I. The *effective* multiplicative fading coefficient $d_{\mathrm{D}_i\mathrm{S}_j}$ describing the transmission from source node S_j to destination node D_i is then obtained as the superposition of all paths connecting these nodes in the network graph:

$$d_{\mathrm{D}_i\mathrm{S}_j} = \sum_{\substack{(\mathrm{R}^{(1)}_{k_1},\dots,\mathrm{R}^{(L)}_{k_L}) \\ \in \mathcal{R}_1\times\dots\times\mathcal{R}_L}} h_{\mathrm{R}^{(1)}_{k_1}\mathrm{S}_j} h_{\mathrm{R}^{(2)}_{k_2}\mathrm{R}^{(1)}_{k_1}} \cdots h_{\mathrm{D}_i\mathrm{R}^{(L)}_{k_L}} g_{\mathrm{R}^{(1)}_{k_1}} \cdots g_{\mathrm{R}^{(L)}_{k_L}}. \tag{1}$$

We aim to identify sets of complex gain coefficients which ensure that each destination node D_i receives no other signals than those of its associated source node S_i. That is, we want to suppress spatial interference between the n source-destination links by *multihop distributed beamforming*. In this context "distributed" refers to the constraint that beamforming must be performed solely based on a multiplication of the received signal by a complex scalar at each relay (as opposed to a matrix multiplication of the vector of signals received in the whole relay stage). In mathematical terms, interference free communication between all pairs $\{\mathrm{S}_i, \mathrm{D}_i\}_{i=1}^n$ (network orthogonalization), is established if

$$d_{\mathrm{D}_i\mathrm{S}_j} = 0 \text{ for all } (\mathrm{D}_i, \mathrm{S}_j) \in \mathcal{D} \times \mathcal{S} \text{ s.t. } i \neq j, \tag{2}$$

$$d_{\mathrm{D}_i\mathrm{S}_i} \neq 0 \text{ for all } i \in \{1, \dots, n\}. \tag{3}$$

Considering the left hand sides of these conditions as multivariate polynomials in the $g_{\mathrm{R}_k^{(l)}}$, orthogonalization of the network is possible, whenever the polynomials $d_{\mathrm{D}_i\mathrm{S}_j}$, $i \neq j$, have at least a single common root that is not a root of any of the polynomials $d_{\mathrm{D}_i\mathrm{S}_i}$.

We remark that solving the equation system (2) requires global channel state information, i.e. the knowledge of *all* fading coefficients h_{IJ}.

B. Existence of Solutions

In the single relay stage case ($L = 1$) the equation system (2) is a linear one, and therefore well analyzable by standard linear algebra methods [4]. In the general case of arbitrary L, we are confronted with a multi-linear polynomial equation system. Whether (and how many) solutions to such systems exist is well understood in the case that the system has generic coefficients. In this context, "generic" – loosely speaking – means that all coefficients of the monomials are independent parameters. It is a seminal result from algebraic geometry, that the number of solutions is then fully determined by the structure of the monomials for almost all sets of coefficients [5]. In the case at hand the $n(n-1)\prod_l n_\mathcal{R}^{(l)}$ monomial coefficients depend on $n \cdot n_\mathcal{R}^{(1)} + \sum_{l=1}^{L-1} n_\mathcal{R}^{(l)} n_\mathcal{R}^{(l+1)} + n_\mathcal{R}^{(L)} \cdot n$ fading coefficients only, thus being subject to a certain structure themselves. This issue renders proving general conditions on sets $\{n_\mathcal{R}^{(l)}\}_{l=1}^L$ for a network with n source-destination pairs being orthogonalizable difficult.

In this paper we restrict ourselves to stating the following conjecture, which is in line with numerical experiments: Assuming all fading coefficients to be distributed according to a continuous probability distribution, the subsequent conditions are necessary and sufficient for orthogonalization in the multihop network with n source-destination pairs being possible with probability one:

$$\sum_{l=1}^{L} n_\mathcal{R}^{(l)} \geq n(n-1) + L \tag{4}$$

$$n_\mathcal{R}^{(l)} \geq n \qquad \forall l \in \{1, \dots, L\}. \tag{5}$$

For $L = 1$, condition (4) reduces to the well-known necessary and sufficient condition $n_\mathcal{R} \geq n(n-1) + 1$ for two-hop networks while condition (5) is redundant [4].

In full generality, we can neither prove sufficiency of both conditions nor necessity of the first condition. The second condition is necessary, in fact: The source transmit vector is seen at the destination antennas under the linear transformation determined by the matrix

$$\mathbf{D} = \mathbf{H}_{L+1}\mathbf{G}_L\mathbf{H}_L \cdots \mathbf{G}_1\mathbf{H}_1, \tag{6}$$

where

$$\mathbf{D} = \left(d_{\mathrm{D}_i\mathrm{S}_j}\right)_{i=1,\ldots,n;j=1,\ldots,n},$$

$$\mathbf{G}_l = \mathrm{diag}\left(\left(g_{\mathrm{R}_k^{(l)}}\right)_{k=1,\ldots,n_{\mathcal{R}}^{(l)}}\right),$$

$$\mathbf{H}_l = \begin{cases} \left(h_{\mathrm{R}_i^{(1)}\mathrm{S}_j}\right)_{i=1,\ldots,n_{\mathcal{R}}^{(1)};j=1,\ldots,n} & \text{if } l=1, \\ \left(h_{\mathrm{R}_i^{(l)}\mathrm{R}_j^{(l-1)}}\right)_{i=1,\ldots,n_{\mathcal{R}}^{(l)};j=1,\ldots,n_{\mathcal{R}}^{(l-1)}} & \text{if } 1<l\leq L, \\ \left(h_{\mathrm{D}_i\mathrm{R}_j^{(L)}}\right)_{i=1,\ldots,n,j=1,\ldots,n_{\mathcal{R}}^{(L)}} & \text{if } l=L+1. \end{cases}$$

In order to fulfill (2) and (3) simultaneously, $\mathbf{D}$ must be a diagonal matrix with non-zero entries on the diagonal. Accordingly, its rank $\mathrm{rk}\{\mathbf{D}\}$ must be n. We conclude that (5) is necessary, since

$$\mathrm{rk}(\mathbf{D}) \leq \min\left(\min_l(\mathrm{rk}\{\mathbf{G}_l\}), \min_l(\mathrm{rk}\{\mathbf{H}_l\})\right) \leq \min_l(\mathrm{rk}\{\mathbf{G}_l\}).$$

For condition (4), let us first define the $\sum_l n_{\mathcal{R}}^{(l)}$-tuple

$$g \triangleq \left(g_{\mathrm{R}_1^{(1)}}, \ldots, g_{\mathrm{R}^{(1)}_{n_{\mathcal{R}}^{(1)}}}, \ldots, g_{\mathrm{R}_1^{(L)}}, \ldots, g_{\mathrm{R}^{(L)}_{n_{\mathcal{R}}^{(L)}}}\right).$$

We realize that whenever the system (2) has a solution g^*, then there must exist infinitely many solution constituting an L-dimensional affine variety: Let $c_1, \ldots, c_L$ be arbitrary, non-zero complex scalars and define $\tilde{g}_{\mathrm{R}_k^{(l)}} \triangleq c_l g_{\mathrm{R}_k^{(l)}}$. Then, also $\tilde{g}^*$ fulfills (2). This is seen by inspecting (1), which reveals the relation

$$d_{\mathrm{D}_i\mathrm{S}_j}\Big|_{g=\tilde{g}^*} = c_1 \cdots c_L \cdot d_{\mathrm{D}_i\mathrm{S}_j}\Big|_{g=g^*} \quad \text{for all } i \neq j. \tag{7}$$

The left hand side of this equation is zero if and only if the right hand side is zero. By the same reasoning, we can say that g^* fulfills (3) if and only if also $\tilde{g}^*$ fulfills (3).

Due to property (7) the considered system (2) is said to be *L-homogeneous* (homogeneous in the L groups of gain coefficients) with multi-degree $(1, \ldots, 1)$ or also *L-linear* (since the c_l occur to the first power on the right hand side of (7)). While such systems have either no or infinitely many solutions in $\mathbb{C}^{\sum_l n_{\mathcal{R}}^{(l)}}$, they can have a finite number of solutions in the product of projective spaces[1] $\mathbb{P}^{n_{\mathcal{R}}^{(1)}-1} \times \ldots \times \mathbb{P}^{n_{\mathcal{R}}^{(L)}-1}$ (e.g. [6]). It is noteworthy, that in affine space the c_l are (up to phase shifts) determined nevertheless if a sum-power constraint is imposed on each stage.

Assuming that no equation or set of equations is implied by any other set of equations, (4) has either no or finitely many solutions in the product of projective spaces if the system has L more unknowns than equations, i.e. if condition (4) holds with equality. In this particular case a dehomogenized system (e.g. with $g_{\mathrm{R}_1^{(l)}} = 1$ for $l \in \{1, \ldots, L\}$) has as many degrees of freedom as equations. Whether solutions indeed do exist depends on whether the coefficients are generic enough (cf. Section II-C), i.e. on whether the coefficients depend on sufficiently many parameters in a sufficiently unstructured way. We assume that this is the case as long as condition (5) holds.

If it is true that there are finitely many solutions to (2), when (4) holds with equality and (5) is fulfilled, reducing the number of unknowns (total number of relays) renders the system overconstrained and does not allow for any solution. Vice versa, if the number of relays is increased the system is underconstrained and there are infinitely many solutions lying on a non-zero dimensional projective variety in the product of projective spaces. This leads to the conjecture that conditions (4) and (5) are necessary and sufficient in fact.

Generally, a necessary and sufficient condition for the existence of a solution is given by the projective weak Nullstellensatz [6]. The problem with this approach is that it requires the construction of a reduced Groebner basis, which is difficult to obtain for general networks.

C. Number of Solutions

Given the conjecture that there exist finitely many solutions to (2) if condition (4) holds with equality, we can give an upper bound on the actual number of solutions. This bound is due to D.N. Bernstein [5] who linked the number of solutions of a system of polynomial equations to the structure of the Newton polytopes of the polynomials. A Newton polytope is defined as follows:

Definition. *Consider a multivariate polynomial $p(z_1, \ldots, z_m) = \sum_\alpha c_\alpha \prod_{i=1}^m z_i^{\alpha_i}$. The Newton polytope of p, denoted by Δ_p, is defined as the convex hull of the set of exponents α, considered as vectors in $\mathbb{Z}^m$.*

By inspecting (1), we realize that all $d_{\mathrm{D}_i\mathrm{S}_j}$ share the same Newton polytope. Such systems are said to be *unmixed* and allow to use a corollary to Bernsteins' general result [7]:

Theorem (Kushnirenko)**.** *If m polynomials $p_1, p_2, \ldots, p_m$ with identical Newton polytope have a finite number of joint zeros in $(\mathbb{C}\backslash 0)^m$, their number is upper bounded by $m!\mathrm{Volume}(\Delta_p)$. The bound holds with equality for generic coefficients.*

Note that the quantity $m!\mathrm{Volume}(\Delta_p)$ always evaluates to an integer value. Specifically, in the case at hand it can be shown that

$$(n^2-n)!\mathrm{Volume}\left(\Delta_{d_{\mathrm{D}_i\mathrm{S}_j}}\right) = \frac{(n^2-n)!}{\prod_{l=1}^L (n_{\mathcal{R}}^{(l)}-1)!}.$$

The fact that solutions containing zero gain coefficients are not accounted for is uncritical. In case such a solution would exist, condition (4) could be relaxed, since relays with zero gain coefficient can be considered as not being present in the network.

An important conclusion from this theorem is that the number of solutions to a polynomial equation system is fully determined by the structure of the monomials for generic coefficients. As discussed in the previous section, the coefficients in (1) depend on a number of parameters which is significantly

[1] The projective space $\mathbb{P}^k$ is the set of all k dimensional lines in $\mathbb{C}^{k+1}$ passing through the origin.

TABLE I

NUMBER OF ZERO-FORCING SOLUTIONS FOR SEVERAL NETWORKS.

Configuration	Solutions	Upper-Bound
$(n, n(n-1)+1)$	1	1
$(2, 2, 2)$	2	2
$(3, 5, 3)$	6	15
$(3, 3, 5)$	6	15
$(3, 4, 4)$	12	20
$(3, 3, 3, 3)$	18	90
$(4, 7, 7)$	≥ 528	924
$(4, 5, 5, 5)$	n/a	34650
$(4, 4, 4, 4, 4)$	n/a	369600

smaller than the number of coefficients. According to the bound of the theorem this lack of genericity can only lead to a reduction of the number of solutions.

In Tab. I we show for a couple of configurations $(n, n_{\mathcal{R}}^{(1)}, \ldots, n_{\mathcal{R}}^{(L)})$ how the number of actual solutions (as identified by numerical solvers) compares to the upper bound. We observe that both the actual number of solutions and the upper bound increase rapidly both in L and in n. The upper-bound holds with equality in the cases $L = 1$ and $L = 2$. For larger L the bound is loose in general. We observe, that distributing relays uniformly over the stages seems to yield more solutions than asymmetric stage allocations.

III. "PING-PONG" IN INTERFERENCE NETWORKS

The orthogonalization of multihop networks as studied in the previous section inspires a new approach to orthogonalizing interference networks with a source and a destination cluster in single-hop distance. A set of source nodes can communicate to a set of destination nodes in an interference free fashion on the same physical channel if both sets of nodes transmit their signals forth and back several times (cf. left hand sketch in Fig. 2). E.g., consider a set of n source nodes that wish to communicate to n destination nodes. Let us denote the fading matrix describing the MIMO channel between source and destination nodes by $\mathbf{H} \in \mathbb{C}^{n\times n}$. Assuming channel reciprocity, the fading matrix describing the channel from destination to source nodes is then given by $\mathbf{H}^T$. In the previous section we found that a multihop relay network with n relay stages containing n relays each can be orthogonalized, since both (4) and (5) are fulfilled with equality. Accordingly, we might expect to be able to orthogonalize n (assumed to be an even number) source-destination pairs by transmitting $n+1$ times forth and back. The effective fading matrix (cf. (6)) would then write as:

$$\mathbf{D} = \mathbf{H}\mathbf{G}_n\mathbf{H}^T\mathbf{G}_{n-1}\cdots\mathbf{H}\mathbf{G}_2\mathbf{H}^T\mathbf{G}_1\mathbf{H},$$

where all matrices $\mathbf{G}_n$ with odd indices correspond to amplify-and-forward operations at the destination nodes and those with even indices to amplify-and-forward operations at the source nodes. For this particular setting numerical experiments suggest that orthogonalization is indeed feasible. Also, the numbers of solutions obtained are in line with those obtained in the equivalent multihop networks. This is far from being evident in fact, since the coefficients in (1) are even less generic in this example: Since each hop is associated with the same channel matrix $\mathbf{H}$ or its transposed, the randomness in the network is reduced from $(L+1)\cdot n^2$ independent random variables (parameters) in the multihop network case to n^2 independent random variables. Therefore, one might have expected that also the number of solutions is reduced.

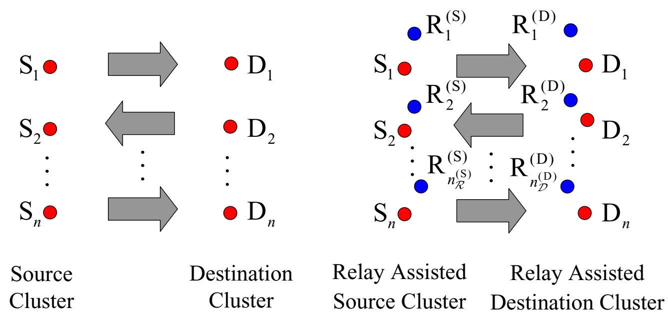

Fig. 2. Mimicking Multihop Networks.

The above example was of little practical value: We managed to orthogonalize n data streams by using $n+1$ time slots. This corresponds to a spatial multiplexing gain of $n/(n+1) < 1$. Accordingly, a simple time-division multiple access scheme outperforms our scheme. Nevertheless, we can render this approach beneficial. In order to achieve spatial multiplexing gains larger than one, the source and/or the destination cluster need to be assisted by additional relay nodes (cf. right hand sketch in Fig. 2). In doing so, we can achieve network orthogonalization with a reduced number of forth and back transmissions N (an odd number ≥ 3), thus increasing the spatial multiplexing gain n/N.

Consider for example a network of four source-destination pairs, where source and destination cluster are assisted by three additional relay nodes each. Indeed, this network can be orthogonalized by transmitting three times forth and back. Accordingly, we achieve a spatial multiplexing gain of 4/3.

Sticking to the strategy of distributing the additional relay nodes equally over the source and destination cluster ($n_{\mathcal{R}}$ per cluster), our approach achieves a spatial multiplexing gain n/N according to (4) as long as

$$n_{\mathcal{R}} \geq \frac{n(n-1)}{N-1} + 1 - n.$$

Numerical experiments suggest that for symmetric relay allocations the number of solutions to (2) is sustained despite of the reduced genericity.

Let us finally consider an asymmetric relay node allocation over the source and destination cluster. For simplicity, we assume a network of three source-destination pairs with two additional relay nodes in the destination cluster, and no additional relay nodes in the source cluster. Again, condition (4) is fulfilled with equality, and also condition (5) holds. In this example, however, we observe that no solution is found by our numerical solver. Here, indeed the effect of lacking genericity in the polynomial coefficients kicks in. Note that this problem is circumvented if the forth and back transmissions are performed over different subcarriers in a frequency selective environment. Then, each transmission is again associated with a different fading matrix, and from an analysis point of view one is back to the multihop network from the previous section.

Ref. [8] provides a coding scheme that achieves a spatial multiplexing gain of $n/2$ in an n user interference network without making use of any relay nodes. Although the scheme outlined above is inferior to this scheme both in terms multiplexing gain and transmit power efficiency, its beauty lies in the much reduced coding complexity. E.g., In a cellular environment, the scheme can be employed for mitigating multi-cell interference. Assuming a cluster of base stations and a cluster of mobile stations (each cluster including some idle stations acting as relays), (matrix) channel estimation and exchange of channel state information is only required on base station side. The latter might be handled via a backbone network. Once the base stations have determined the gain coefficients, only the gain coefficients of the mobile stations need to be disseminated over the wireless channel. Moreover, in the symmetric case only a single fading matrix has to be learned on base station side. The scheme is suitable both for up- and downlink transmission.

IV. Experimental Insights

In this section we discuss the following two issues by means of simulation results: First, we are interested in the performance of the different zero-forcing solutions, in particular in the question to which extend performance varies under different solutions. Second, we want to understand the impact of the dimensions of the network, in particular how performance is influenced by the length (L) of the network. Our measure of performance is achievable rate. Since each source-destination pair communicates over an additive white Gaussian noise channel for a given fading realization, the rate of source-destination pair i under zero-forcing solution j is given by

$$R_i^{(j)} = \log(1 + \mathrm{SNR}_i^{(j)}), \tag{8}$$

where the $(L+1)^{-1}$ pre-log factor is discarded[2] and

$$\mathrm{SNR}_i^{(j)} = \frac{P_{\mathcal{S}}/n}{\sigma^2} \cdot \frac{\left|d_{\mathrm{D}_i\mathrm{S}_i}\right|^2}{1 + \sum_{l,k_l} \left|d_{\mathrm{D}_i\mathrm{R}_{k_l}^{(l)}}\right|^2} \tag{9}$$

is the corresponding signal-to-noise ratio. Here, we denote by $P_{\mathcal{S}}$ the source cluster transmit power, which is allocated uniformly over the source nodes, and by σ^2 the variance of the i.i.d. zero-mean additive white Gaussian noise introduced by each receiving node. Finally, $d_{\mathrm{D}_i\mathrm{R}_k^{(l)}}$ denotes the effective fading coefficient seen from relay node $\mathrm{R}_k^{(l)}$ to destination node D_i. We are particularly interested in the achievable sum-rate $R_\Sigma^{(j)} \triangleq \sum_i R_i^{(j)}$ and in the rate achievable for the weakest source-destination pair $R_{\min}^{(j)} \triangleq \min_i R_i^{(j)}$ (referred to as minimum-rate). We generate realizations of the fading coefficients h_{IJ} independently from a circular symmetric complex Gaussian random variable of zero mean and unit variance.

[2] We intentionally drop this multiplicative constant in order to eliminate its influence on rates achievable in networks of different length. In practice the pre-log is not necessarily $(L+1)^{-1}$, but typically smaller and independent of L. It depends on path loss and shadowing effects, which (if strong enough) allow for injecting new signals into the network more often than every $L+1$ time slots without posing interference to previously injected signals.

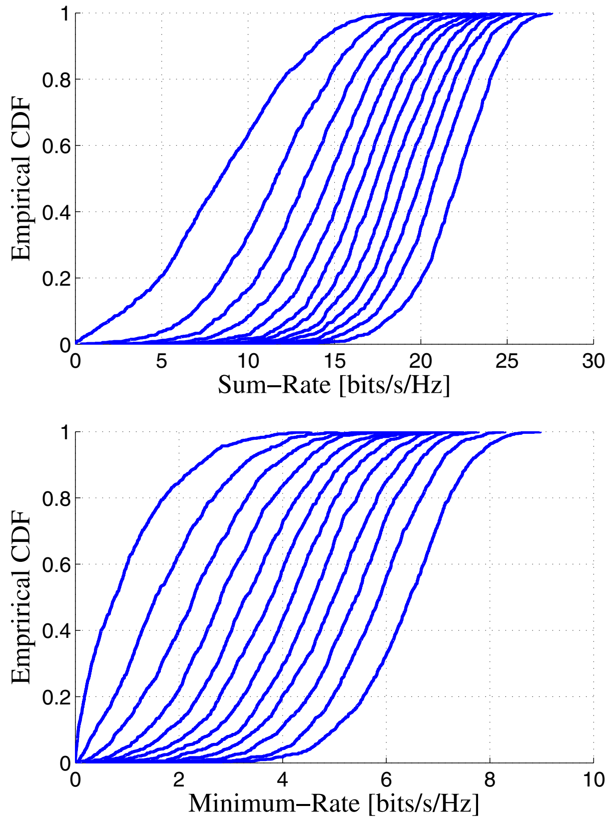

Fig. 3. Empirical CDFs of order statistics of sum-rates $R_\Sigma^{(1)}, \ldots, R_\Sigma^{(12)}$ and minimum-rates $R_{\min}^{(1)}, \ldots, R_{\min}^{(12)}$ for a $(n, n_{\mathcal{R}}^{(1)}, n_{\mathcal{R}}^{(2)}) = (3, 4, 4)$-network.

In the following we fix $P \triangleq P_{\mathcal{S}} = P_{\mathcal{R}_1} = \ldots = P_{\mathcal{R}_L} = 1000 \cdot (L+1)$, where $P_{\mathcal{R}_l}$ denotes the sum-transmit power of stage $\mathcal{R}_l$, and $\sigma^2 = 1$. This power allocation results in a destination SNR of about 30 dB for small L, when all relay nodes share the common gain coefficient $\sqrt{(P/n_{\mathcal{R}}^{(l)})/(P+1)}$.

A. Comparison of Solutions

We consider a network with three source-destination pairs and two relay stages containing four relays each. This network exhibits twelve different zero-forcing solutions. We conduct the following experiment: For 1000 channel realizations, we determine all twelve zero-forcing solutions numerically, and thereupon evaluate the corresponding rates $R_\Sigma^{(j)}$ and $R_{\min}^{(j)}$ for all $j \in \{1, \ldots, 12\}$. We obtain empirical cumulative distribution functions (CDF) of the order statistics of $R_\Sigma^{(j)}$ and $R_{\min}^{(j)}$. The respective plots are shown in Fig. 3. The key conclusion to be drawn from these plots is that both for $R_\Sigma^{(j)}$

and $R_{\min}^{(j)}$ there are tremendous differences in performance. The average achievable rate obtained for the best solution is around five times larger than the one obtained for the worst solution in terms of minimum-rate and still more than twice larger in terms of sum-rate. While finding the best zero-forcing solution based on a brute force search is manageable for the case at hand with only twelve solutions, such an approach seems to be hopeless for larger networks where the number of solutions grows rapidly (cf. Section II-C).

B. Impact of Network Dimensions

There are various references pointing out, that increasing the length of amplify-and-forward multihop networks reduces the achievable sum-rate even if the destination signal-to-noise ratio is kept constant an the pre-log factor is neglected [9], [10], [11]. These works considered a fully cooperating-destination cluster (joint MIMO decoding) and relay gain matrices $\mathbf{G}_l \propto \mathbf{I}$, which is a reasonable choice if global channel state information is not available in all stages. The effect such networks suffer from, is an undesired distortion of the eigenvalue spectrum of the effective fading matrix $\mathbf{D}$. We raise the question, whether such an effect can be observed in the setting considered in this paper as well. We find an answer by inspecting Fig. 4: Here we show empirical CDFs of the order statistics of the n source-destination pairs under the sum-rate optimal zero-forcing solution for a network with three source-destination pairs. We compare networks with $L = 1$ and $n_{\mathcal{R}}^{(1)} = 7$, $L = 2$ and $n_{\mathcal{R}}^{(1)} = n_{\mathcal{R}}^{(1)} = 4$, as well as $L = 3$ and $n_{\mathcal{R}}^{(1)} = n_{\mathcal{R}}^{(2)} = n_{\mathcal{R}}^{(3)} = 3$. We observe, that indeed the longest network demonstrates worst performance. However, surprisingly, the three-hop network is superior over the two-hop network. This behavior identifies an effect, which is antipodal to the effect described in the references mentioned above: increasing the network length increases the number of zero-forcing solutions. Optimizing over these solutions allows for compensating the eigenvalue distortion to a certain extend. Note that other than the networks studied in [10], [11], our power allocation does not ensure a constant average signal-to-noise ratio at the destination nodes, since the matrices $\mathbf{G}_l$ are not proportional to the identity matrix in general.

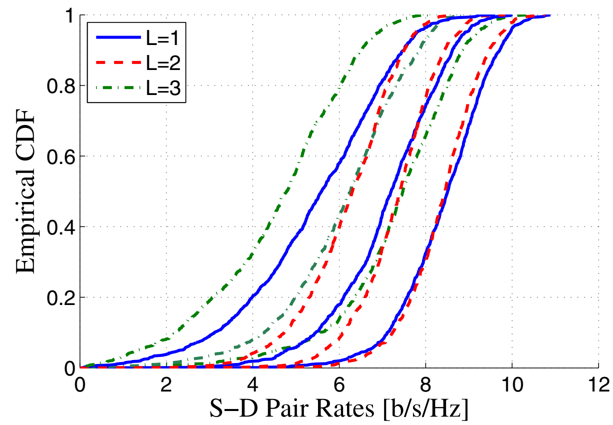

Fig. 4. Empirical CDFs of order statistics of $R_1^{(j)}, \ldots, R_3^{(j)}$, $j = \operatorname{argmax}_j R_\Sigma^{(j)}$, for $(3,7)$, $(3,4,4)$ and $(3,3,3,3)$ networks.

V. Conclusions

We have generalized the concept of distributed zero-forcing to networks with an arbitrary number of hops. Interestingly, the total number of relays required for rendering the network orthogonalizable increases only by one per additional hop. Thus, the required number of relays per stage decreases significantly at the same time. Moreover, the results obtained on multihop networks inspired a new approach for orthogonalization of single-hop interference networks. The multilinear structure of the equation system to be solved in the process of obtaining suitable relay gain coefficients poses several difficulties when compared to two-hop networks: Especially, in the case that there are more relays in the network than stringently needed (as treated in the two-hop case in [12]) optimization of the gain coefficients under a zero-forcing constraint strikes us as becoming a challenging task. Finally, our computer experiments suggest that the common understanding that increasing the length of amplify-and-forward networks penalizes achievable sum-rates is not valid in this generality, when relays can determine their gain coefficients based on global channel state information.

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