On the Capacity of Relay-Assisted Wireless MIMO Channels

Boris Rankov and Armin Wittneben
Swiss Federal Institute of Technology (ETH) Zurich
Communication Technology Laboratory
Sternwartstrasse 7, CH-8092 Zurich, Switzerland
Email: {rankov, wittneben}@nari.ee.ethz.ch

Abstract — **We study the impact of multiple amplify-and-forward relays on the capacity of wireless MIMO channels. For wireless networks with one source/destination pair (equipped with multiple antennas) and several single-antenna amplify-and-forward relays we determine the compound (over two time slots) channel matrix of the relay-assisted MIMO channel. We derive the asymptotic eigenvalues of the compound channel matrix by letting the number of transmit and receive antennas go to infinity (large-array limit). Then the asymptotic ergodic capacity of the amplify-and-forward transmission scheme is determined for Rayleigh and Rice fading. The results serve as tight approximation for a finite number of transmit and receive antennas and are accurate in the large-array limit.**

I. Introduction

Multiple antennas at transmitter and receiver introduce spatial degrees of freedom into a wireless communication system. Space-time signal processing utilizes these degrees of freedom to boost link capacity and/or to enhance link reliability of multiple-input multiple-output (MIMO) communication systems. With *spatial multiplexing* one can increase the data rate without additional cost of bandwidth or power by transmitting data streams simultaneously over spatial sub-channels which are available in a rich scattering environment [1].

It is expected that future wireless broadband communication systems will operate beyond 5 GHz, for example Wireless Local Area Networks (WLANs) at 17 GHz (Hiperlan) or at 24/60 GHz (ISM bands). In higher frequency bands it is possible to accommodate a larger number of antennas in a given volume ("rich array") because the array size scales down with increasing frequency. Further on, the array gain of the system can compensate the path loss which is inversely proportional to the square of the frequency [2].

For zero-mean i.i.d. Gaussian channel coefficients the ergodic capacity of a MIMO channel with M transmit and N receive antennas scales linearly with $\min\{M,N\}$ compared to a corresponding single-input single-output (SISO) channel. However, there is a major obstacle in the practical exploitation of MIMO technology: the capacity gain depends strongly on the propagation environment and diminishes with increasing correlation of the channel coefficients [3]. In higher frequency bands we expect an increase in correlation because the propagation channel becomes more and more line-of-sight (LOS) and we are confronted with a *rich array – poor scattering regime* [2].

In [4] it is shown that with cooperative two-hop relaying one can increase the rank and therefore the capacity of correlated (ill-conditioned) MIMO channels. The main idea is to use *amplify-and-forward relays* that act as active scatterers and assist the communication between source and destination: The relays receive in the first time slot the signal from the source and forward an amplified version to the destination in the second time slot. This way of relaying leads to low-complexity relay transceivers and to lower power consumption since there is no signal processing for decoding procedures. The goal of cooperative relaying here is to increase the rank of the compound (two time slots) channel matrix and to shape the eigenvalue distribution such that the channel matrix becomes well-conditioned and the capacity of the MIMO channel improves.

Related Work. Laneman et al. develop and analyze in [5] cooperative diversity protocols for combating multipath fading in a wireless network. The protocols exploit spatial diversity available among a collection of distributed terminals that relay messages for one another. The authors compare the performance of decode-and-forward and amplify-and-forward relay strategies with respect to outage capacity. In [6] Sendonaris et al. investigate a form of spatial diversity, in which diversity gains are achieved via the cooperation of two mobile users that communicate with a base station. It is shown that cooperation leads not only to an increase in uplink capacity for both users but also to a more robust system, where user's rates are less sensitive to channel variations.

General capacity results on relay channels have been found in the seventies by Cover and Gamal in [7] and the references therein. More recently upper and lower bounds on the capacity of wireless networks with a relay traffic pattern have been determined by Gastpar and Vetterli in [8]. In this paper the system model consists of one source/destination pair, while all other nodes operate as relays in order to assist this transmission. To achieve capacity the nodes use arbitrary complex network coding. In [9] Nabar et al. extended the analysis of [8] and gave upper and lower bounds on the capacity of MIMO wireless networks with a relay traffic pattern.

Organization of the Paper. Section II describes the idea of the cooperative MIMO system and introduces the signal model. In section III we derive the eigenvalues of the relay-assisted MIMO channel and give the corresponding capacity expressions for Rayleigh and Rice fading. Section IV presents and discusses numerical results. Conclusions are given in the last section.

Notation. We use bold upper letters to denote matrices and bold lower letters to denote vectors. Further $(\cdot)^*$, $(\cdot)^{\mathrm{T}}$, $(\cdot)^{\mathrm{H}}$ stand for conjugation, transposition and Hermitian transposition, respectively. $\mathrm{E}\{\cdot\}$ denotes the expectation operator, $\mathrm{rk}(\cdot)$ the rank and $\lambda_k(\cdot)$ the kth eigenvalue of a matrix. $\mathbf{I}_n$ is an $n\times n$ identity matrix. We shall use $|\cdot|$ to denote the magnitude of a complex scalar and $\|\cdot\|_{\mathrm{F}}$ to denote the Frobenius norm of a matrix. With $[\cdot]_{n,k}$ we denote an element from the nth row and kth column of a matrix. Here a circularly symmetric complex Gaussian random variable Z is a random variable $Z = X + jY \sim \mathcal{CN}(m, \sigma^2)$, where X and Y are i.i.d. $\mathcal{N}\left(m, \frac{\sigma^2}{2}\right)$. Throughout this paper we use complex baseband notation.

II. System and Signal Model

Fig. 1 illustrates the proposed cooperative MIMO system. The source (Tx) and the destination (Rx) have both N antennas. K relays assist the communication between source and destination. In order to keep the implementation complexity of the relay nodes low we assume that the relays are single antenna nodes. The transmission of a data packet from the source to the destination occupies two time slots. The relays receive during the first time slot and retransmit an amplified version of the received signal during the second time slot. We assume here that the relays cannot transmit and receive simultaneously and that all nodes are perfectly synchronized.

For the uniform linear array depicted in Fig. 1 we denote the steering vector with respect to ϕ_k as

$$\mathbf{\Phi}_k = \left[1, e^{j2\pi d \sin\phi_k/\lambda}, \ldots, e^{j2\pi(N-1)d\sin\phi_k/\lambda}\right]^{\mathrm{T}}, \quad (1)$$

which follows from a narrowband signal and planar wavefront assumption [10]. The array response for a plane wave arriving at angle θ_k follows accordingly as

$$\mathbf{\Theta}_k = \left[1, e^{-j2\pi d \sin\theta_k/\lambda}, \ldots, e^{-j2\pi(N-1)d\sin\theta_k/\lambda}\right]^{\mathrm{T}}, \quad (2)$$

where d is the antenna separation at source and destination, λ the operational wavelength and $\phi_k, \theta_k \in [-\pi/2, \pi/2]$ the horizontal angles characterizing the paths to and from relay k, respectively (Fig. 1).

The signal received in the first time slot by relay k is

$$y_{\mathrm{R},k} = h_{\mathrm{u},k}\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{s}_1 + n_{\mathrm{R},k}$$

and by the destination

$$\mathbf{y}_1 = \mathbf{H}_{\mathrm{SD}}\mathbf{s}_1 + \mathbf{n}_{\mathrm{D}},$$

where $\mathbf{H}_{\mathrm{SD}}$ is the $N \times N$ channel matrix of the direct MIMO channel (source to destination), $h_{\mathrm{u},k}$ the relay-uplink (source to relay) channel coefficient of relay k which usually accounts for path loss, shadowing and small-scale fading and $\mathbf{y}_1 = [y_{1,1}, \ldots, y_{1,N}]^{\mathrm{T}}$ is the receive vector at the destination in time slot 1. We assume that the fading coefficient is approximately the same for all transmit antennas (spacing of antenna elements at both sides sufficiently small), i.e., $h_{\mathrm{u},k}^{(1)} \approx h_{\mathrm{u},k}^{(2)} \approx \ldots \approx h_{\mathrm{u},k}^{(N)} = h_{\mathrm{u},k}$, where $h_{\mathrm{u},k}^{(j)}$ denotes the channel coefficient between transmit antenna j and relay antenna k. $n_{\mathrm{R},k}$ and $\mathbf{n}_{\mathrm{D}} = [n_{\mathrm{D},1}, \ldots, n_{\mathrm{D},N}]$ describe the complex-valued zero-mean additive white Gaussian noise (AWGN) at relay k and destination with per-antenna variance σ_{R}^2 and σ_{D}^2, respectively. The vector $\mathbf{s}_1$ with $\mathrm{E}\{\mathbf{s}_1^{\mathrm{H}}\mathbf{s}_1\} = P$ is the source signal in the first time slot. In the second time slot the destination receives

$$\mathbf{y}_2 = \sum_{k=1}^{K} h_{\mathrm{d},k} g_k h_{\mathrm{u},k} \mathbf{\Theta}_k \mathbf{\Phi}_k^{\mathrm{T}}\mathbf{s}_1 + \sum_{k=1}^{K} h_{\mathrm{d},k} g_k \mathbf{\Theta}_k n_{\mathrm{R},k} + \mathbf{H}_{\mathrm{SD}}\mathbf{s}_2 + \mathbf{n}_{\mathrm{D}},$$

where $h_{\mathrm{d},k}$ is the relay-downlink (relay to destination) channel coefficient of relay k (again approximately the same for all receive antennas), g_k is the gain factor of relay k and $\mathbf{y}_2 = [y_{2,1}, \ldots, y_{2,N}]^{\mathrm{T}}$ is the receive vector at the destination in time slot 2. Further we have $\mathbf{s}_2$ with $\mathrm{E}\{\mathbf{s}_2^{\mathrm{H}}\mathbf{s}_2\} = P$ which is the source signal in the second time slot. In matrix notation we write

$$\mathbf{y}_2 = \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{ud}}\mathbf{\Phi}^{\mathrm{T}}\mathbf{s}_1 + \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{d}}\mathbf{n}_{\mathrm{R}} + \mathbf{H}_{\mathrm{SD}}\mathbf{s} + \mathbf{n}_{\mathrm{D}},$$

where $\mathbf{\Gamma}_{\mathrm{d}}$ and $\mathbf{\Gamma}_{\mathrm{ud}}$ are diagonal with $[\mathbf{\Gamma}_{\mathrm{ud}}]_{k,k} = h_{\mathrm{d},k} g_k h_{\mathrm{u},k}$ and $[\mathbf{\Gamma}_{\mathrm{d}}]_{k,k} = h_{\mathrm{d},k} g_k$. The columns of $\mathbf{\Phi}$ are the steering vectors (1) and the columns of $\mathbf{\Theta}$ are the array response vectors (2) and $\mathbf{n}_{\mathrm{R}} = [n_{\mathrm{R},1}, \ldots, n_{\mathrm{R},K}]$. We stack the receive vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ into one vector and obtain the following description of the two-hop MIMO relay channel

$$\mathbf{y} = \begin{bmatrix} \mathbf{H}_{\mathrm{SD}} & \mathbf{0} \\ \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{ud}}\mathbf{\Phi}^{\mathrm{T}} & \mathbf{H}_{\mathrm{SD}} \end{bmatrix} \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{n}_{\mathrm{D}} \\ \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{d}}\mathbf{n}_{\mathrm{R}} + \mathbf{n}_{\mathrm{D}} \end{pmatrix}$$
$$= \mathbf{H}\mathbf{s} + \mathbf{n}. \quad (3)$$

We may have three different traffic patterns which translates into three different signal models [4]:

Traffic Pattern T1. The source transmits $\mathbf{s}_1$ in the first time slot to the destination and the relays. In the second time slot the relays forward the signals to the destination and the source transmits simultaneously $\mathbf{s}_2$ to the destination. The signal structure is given in (3).

Traffic Pattern T2. The source transmits $\mathbf{s}_1$ in the first time slot to the destination and the relays. In the second time slot the relays forward the signals to the destination whereas the source is inactive. The signal structure changes to

$$\mathbf{y} = \begin{bmatrix} \mathbf{H}_{\mathrm{SD}} \\ \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{ud}}\mathbf{\Phi}^{\mathrm{T}} \end{bmatrix} \mathbf{s}_1 + \begin{pmatrix} \mathbf{n}_{\mathrm{D}} \\ \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{d}}\mathbf{n}_{\mathrm{R}} + \mathbf{n}_{\mathrm{D}} \end{pmatrix}.$$

Traffic Pattern T3. The source transmits $\mathbf{s}_1$ in the first time slot only to the relays (the direct source-destination link is blocked for example due to shadowing). In the second time slot the relays forward the signals to the destination whereas the source is inactive. The signal structure follows then as

$$\mathbf{y} = \mathbf{y}_2 = \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{ud}}\mathbf{\Phi}^{\mathrm{T}}\mathbf{s}_1 + \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{d}}\mathbf{n}_{\mathrm{R}} + \mathbf{n}_{\mathrm{D}}.$$

Due to space constraints and simplicity of analysis we will consider in this paper mostly traffic pattern **T3**.

III. Asymptotic Eigenvalues and Ergodic Capacity

Mutual Information. First we derive the mutual information for the general traffic pattern **T1**. The mutual information for the other traffic patterns follows immediately. Notice that the noise in (3) is spatially colored with autocovariance matrix

$$\mathbf{R}_{nn} = \begin{bmatrix} \sigma_{\mathrm{D}}^2\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{d}}\mathbf{\Gamma}_{\mathrm{d}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}}\sigma_{\mathrm{R}}^2 + \sigma_{\mathrm{D}}^2\mathbf{I}_N \end{bmatrix}.$$

The mutual information of (3) assuming perfect knowledge of $\mathbf{H}$ and $\mathbf{R_{nn}}$ at destination is given by

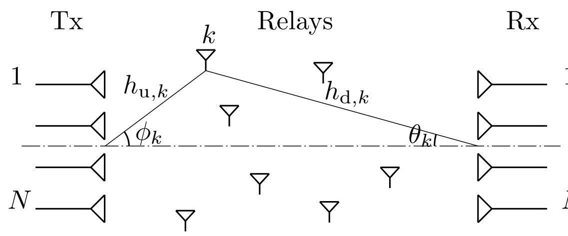

Fig. 1: Relay-assisted MIMO communication system

$I_{\mathrm{T1}}(\mathbf{s};\mathbf{y}|\mathbf{H},\mathbf{R}_{nn}) = h(\mathbf{y}|\mathbf{H},\mathbf{R}_{nn}) - h(\mathbf{n}|\mathbf{R}_{nn})$, where $h(\cdot)$ denotes the differential entropy of a random vector. The mutual information I_{T1} is maximized when $h(\mathbf{y}|\mathbf{H},\mathbf{R}_{nn})$ is maximized, i.e., the receive vector $\mathbf{y}$ has to be Gaussian for a given channel matrix and its differential entropy is [11]

$$h(\mathbf{y}|\mathbf{H},\mathbf{R}_{nn}) = \log_2 \det\left(\pi e\left(\mathbf{H}\mathbf{R}_{ss}\mathbf{H}^{\mathrm{H}} + \mathbf{R}_{nn}\right)\right),$$

where $\mathbf{R}_{ss} = \mathrm{E}\left\{\mathbf{s}\mathbf{s}^{\mathrm{H}}\right\}$. The mutual information measured in bits per channel use follows then with

$$I_{\mathrm{T1}}(\mathbf{s};\mathbf{y}|\mathbf{H}) = \frac{1}{2}\sum_{k=1}^{r}\log_2\left(1 + \frac{P}{N}\lambda_k\left(\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^{\mathrm{H}}\right)\right), \qquad (4)$$

where $r = \mathrm{rk}\left(\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^{\mathrm{H}}\right) = \min\{2N, K\}$. We used $\mathbf{R}_{ss} = \frac{P}{N}\mathbf{I}_{2N}$ since we assume no channel knowledge at source and relays and hence optimal power allocation is not possible, i.e., in every time slot the source distributes the power P equally among the antennas. The factor $\frac{1}{2}$ is due to the use of two time slots.

Asymptotic Eigenvalues. We determine the eigenvalues of the channel correlation matrix $\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^{\mathrm{H}}$ when the number of antennas N goes to infinity (large-array limit) and the antenna separation d remains constant. For the derivation we use a similar approach as in [12]. The eigenvalues are asymptotically accurate as $N \to \infty$ and serve as an approximation in the non-asymptotic regime.

For sake of simplicity we proceed with traffic pattern **T3** where the correlation matrix of the channel in (4) becomes

$$\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^{\mathrm{H}} = \left(\mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{d}}\mathbf{\Gamma}_{\mathrm{d}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}}\sigma_{\mathrm{R}}^2 + \sigma_{\mathrm{D}}^2\mathbf{I}_N\right)^{-1}\mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{ud}}\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}^{\star}\mathbf{\Gamma}_{\mathrm{ud}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}} \qquad (5)$$

with $\mathbf{R}_{nn} = \mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{d}}\mathbf{\Gamma}_{\mathrm{d}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}}\sigma_{\mathrm{R}}^2 + \sigma_{\mathrm{D}}^2\mathbf{I}_N$. In the large-array limit we obtain for the eigenvalues:

Theorem. For $N \to \infty$ the eigenvalues of $\frac{1}{N}\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^{\mathrm{H}}$ are given by

$$\lambda_k\left(\frac{1}{N}\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^{\mathrm{H}}\right) \overset{N\to\infty}{\longrightarrow} \frac{|h_{\mathrm{u},k}|^2}{\sigma_{\mathrm{R}}^2} \qquad (6)$$

for $k = 1,\ldots,K$ and $h_{\mathrm{d},k}, g_k \neq 0\ \forall k$. Here $h_{\mathrm{u},k}$ denotes the kth relay-uplink coefficient and σ_{R}^2 the relay noise variance.

Proof. See the Appendix.

From the derivation of the Theorem we can give an approximation for the eigenvalues [13]:

Corollary. For finite N the eigenvalues of $\frac{1}{N}\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^{\mathrm{H}}$ are approximated by

$$\lambda_k\left(\frac{1}{N}\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^{\mathrm{H}}\right) \approx \frac{N|h_{\mathrm{u},k}g_k h_{\mathrm{d},k}|^2}{\sigma_{\mathrm{D}}^2 + N|g_k h_{\mathrm{d},k}|^2\sigma_{\mathrm{R}}^2} \qquad (7)$$

for $k = 1,\ldots,K$ and where $h_{\mathrm{u},k}$ denotes the kth relay-uplink coefficient, $h_{\mathrm{d},k}$ denotes the kth relay-downlink coefficient and g_k the gain coefficient of relay k.

The theorem and corollary stated above have an interesting physical interpretation: From [10] we know that for a uniform linear array with weighting vector $\mathbf{w} = \frac{P}{N}\mathbf{\Phi}_k$ at the source the 3-dB beamwidth (half-power points) is $\Delta_{3\mathrm{dB}} = 0.891\frac{\lambda}{Nd}$ and the Rayleigh resolution limit (null-to-null beamwidth) is $\Delta_{00} = 2\frac{\lambda}{Nd}$, i.e., the beams from source to relays become narrower with increasing number of transmit antennas and the spatial overlap between the beams disappear. The same holds for the receive side, when $\mathbf{w} = \mathbf{\Theta}_k$ (see Fig. 2). In our case the "beams" (eigenvectors of the channel correlation matrix (5)) become spatially orthogonal in the large-array limit and the kth eigenvalue of the compound channel matrix depends only on the channel parameters of relay k. Actually, the kth eigenvalue depends only on the relay-uplink and the relay noise variance: due to destination's array gain, the signal-to-noise (SNR) ratio at the destination is determined by the relay noise only (since the destination noise can be averaged out through the array gain) and by the relay-uplink coefficient only (relay-downlink is perfectly compensated due to the array gain).

Asymptotic Ergodic Capacity. Due to space constraints we assume here only static relay topologies, i.e., the relays does not change their positions during the time of interest. The channel coefficients in the relay-uplink and relay-downlink are random variables that are constant during one block of transmission. With random coding over a large number of independent blocks one can achieve the ergodic capacity of the system [7].

Combining (4), (5) and (6) we obtain for the asymptotic mutual information under traffic pattern **T3**:

$$I_{\mathrm{T3}}^{\infty}(\mathbf{s};\mathbf{y}|\mathbf{H}) = \frac{1}{2}\sum_{k=1}^{K}\log_2\left(1 + \rho|h_{\mathrm{u},k}|^2\right),$$

where $\rho = P/\sigma_{\mathrm{R}}^2$ and $r = K$. In order to determine the asymptotic ergodic capacity, we assume for the relay-uplink coefficients a model that includes path loss and small scale-fading:

$$h_{\mathrm{u},k} = \frac{1}{(1 + r_{\mathrm{u},k})^{\alpha/2}}x_{\mathrm{u},k}. \qquad (8)$$

Here $1 + r_{\mathrm{u},k}$ is the distance between source and relay k, and $x_{\mathrm{u},k} \sim \mathcal{CN}(m,\sigma^2)$.

Rayleigh Fading. We assume i.i.d. $x_{\mathrm{u},k}$ with zero mean $m = 0$, i.e., $|x_{\mathrm{u},k}|^2$ has an exponential probability density function and the asymptotic ergodic capacity is obtained by taking the expectation over the channel statistics

$$\begin{aligned} C_{\mathrm{T3}}^{\infty} &= \frac{1}{2}\sum_{k=1}^{K}\int_0^{\infty}\log_2\left(1 + \frac{\rho|x_{\mathrm{u},k}|^2}{(1 + r_{u,k})^{\alpha}}\right)\frac{1}{2\sigma^2}e^{-\frac{|x_{\mathrm{u},k}|^2}{2\sigma^2}}\mathrm{d}|x_{\mathrm{u},k}|^2 \\ &= \frac{\ln 2}{2}\sum_{k=1}^{K}e^{\frac{1}{\rho_k 2\sigma^2}}\,\mathrm{Ei}\left(-\frac{1}{\rho_k 2\sigma^2}\right) \end{aligned} \qquad (9)$$

with $\rho_k = \rho/(1 + r_{u,k})^{\alpha}$ and where $\mathrm{Ei}(x)$ is the exponential integral defined as $\mathrm{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}\mathrm{d}t$.

Rice Fading. We assume i.i.d. $x_{\mathrm{u},k}$ with non-zero mean m per real dimension, i.e., $z = |x_{\mathrm{u},k}|^2$ has a non-central chi-square distribution with two degrees of freedom and an upper bound on the asymptotic ergodic capacity is then

$$\begin{aligned} C_{\mathrm{T3}}^{\infty} &= \frac{1}{2}\sum_{k=1}^{K}\int_0^{\infty}\log_2(1 + \rho_k z)\frac{1}{2\sigma^2}e^{-\frac{2m^2+z}{2\sigma^2}}J_0\left(\sqrt{2z}\frac{m}{\sigma^2}\right)\mathrm{d}z \\ &\leq \frac{1}{2}\sum_{k=1}^{K}\int_0^{\infty}\log_2(\rho_k z)\frac{1}{2\sigma^2}e^{-\frac{2m^2+z}{2\sigma^2}}J_0\left(\sqrt{2z}\frac{m}{\sigma^2}\right)\mathrm{d}z \\ &= \frac{1}{2\ln 2}\sum_{k=1}^{K}\left(\ln\left(2\rho_k m^2\right) - \mathrm{Ei}\left(2m^2\right)\right). \end{aligned} \qquad (10)$$

where we used in the inequality a high-SNR approximation and in the last equality a result from [14] about the expected-log of a non-central chi-square random variable. $J_n(x)$ denotes the nth-order modified Bessel function of the first kind.

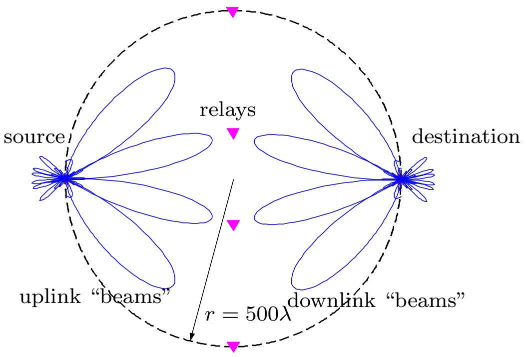

Fig. 2: Source and destination with multiple antennas, relays are single antenna nodes, at 17 GHz we have $r = 500\lambda \approx 8.8$m

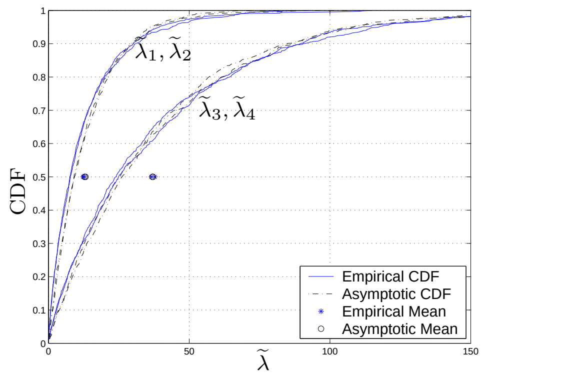

Fig. 3: Eigenvalue CDFs of the relay-assisted MIMO Rayleigh channel for $K = 4$ relays and $N = 20$ antennas, due to symmetrical reasons (Fig. 2) we have $\mathrm{E}\{\lambda_1\} = \mathrm{E}\{\lambda_2\}$ and $\mathrm{E}\{\lambda_3\} = \mathrm{E}\{\lambda_4\}$

IV. Numerical Results

In this section we present numerical examples in order to demonstrate the accuracy of the asymptotic and approximate eigenvalues given in (6) and (7) and the ergodic capacities given in (9) and (10).

Simulation Setup. The setup of the relay network is depicted in Fig. 2. As mentioned before we consider here only deterministic relay positions. In [13] we also consider random relay topologies, i.e., the relays may change their position during the data transmission according to a predefined probability distribution and the capacity results have to be averaged over the relay topology. Here the relays are placed such that the angle difference between two relays with respect to source and destination is constant. This is motivated by the observation that by increasing the angle difference (decreasing the relay density) we obtain less spatial crosstalk between the “beams” (eigenvectors of channel correlation matrix) for finite number of antennas and with that the results in (6) and (7) become more accurate. Note that in the large-array limit the beams become infinitely narrow and no spatial overlap between the beams occur (channel correlation matrix becomes diagonal).

The relay-uplink coefficients are chosen according to (8), where in the Rayleigh case we choose $m = 0$ and $\sigma^2 = 1/2$. For Rice channel simulations we choose $m = \sqrt{K_{\mathrm{R}}/2}$, where K_{R} denotes the Ricean K-factor. The same holds for the relay-downlink coefficients.

Under the assumption that the relays can measure the receive power the gain coefficients in the amplify-and-forward relays are chosen according to $g_k = \sqrt{Q_k/\left(|h_{\mathrm{u},k}|^2 P + \sigma_{\mathrm{R}}^2\right)}$ where $Q_k = P/K$ denotes the maximum transmit power of relay k. Note that this is in general a suboptimal power allocation and other strategies can achieve a better performance [4].

Channel Normalization. In order to obtain defined average SNR values at the destination, we normalize the channel matrix for the simulation such, that the average channel gain is equal to the array gain:

$$\widetilde{\mathbf{H}} = \frac{\mathbf{R}_{nn}^{-1/2}\mathbf{H}}{\sqrt{\mathrm{E}\left\{\|\mathbf{R}_{nn}^{-1/2}\mathbf{H}\|_{\mathrm{F}}^2\right\}}} N. \qquad (11)$$

The total average received power is then $\mathrm{E}\left\{\|\widetilde{\mathbf{H}}\|_{\mathrm{F}}^2\right\} \frac{P}{N} = NP$, i.e., it equals the total transmitted power times the receive array gain N. This implies that the eigenvalues have also to be normalized according to

$$\widetilde{\lambda}_k = \frac{\lambda_k\left(\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^H\right)}{\mathrm{E}\left\{\sum_{k=1}^{K}\lambda_k\left(\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^H\right)\right\}} N^2,$$

where we use for $\lambda_k\left(\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^H\right)$ either (6) or (7). Further we choose an operational frequency of 17GHz , an antenna separation of $d = \lambda/2$ and an average destination SNR of 20dB (averaged over the small-scale fading).

Results. Fig. 3 shows the distribution functions (CDFs) of the non-zero eigenvalues of a relay-assisted MIMO system with $K = 4$ relays and $N = 20$ transmit/receive antennas as example. We see that the CDFs based on the asymptotic eigenvalues are quite close to the corresponding empirical distributions. In Fig. 4 we plot capacity vs. number of antennas assuming Rayleigh fading (in relay-uplink and relay-downlink) for different number of relays. We compare the results obtained via the asymptotic (6) and the approximated (7) eigenvalues with the empirical capacity curve. Capacity scales linearly with number of relays when $K \geq N$, and logarithmic when $K \leq N$ (array gain). Fig. 5 shows the results for Rice fading. A key observation here is, that the capacity is independent of the Ricean factor (note that the effect of an increased receive power due to the LOS component is removed due to our normalization (11)). The rank of the channel matrix and the eigenvalue distribution is determined by the number of relays and their locations. The relays play a role of *active channel shapers* and make the performance of the system insensitive to the small-scale fading statistics.

V. Conclusions

In this paper we gave approximative and asymptotic expressions for the eigenvalues of an AF relay-assisted MIMO channel. Using these eigenvalues we determined the asymptotic ergodic capacity of the amplify-and-forward transmission scheme for Rayleigh and Rice fading under the assumption of a deterministic relay topology. The eigenvalue and capacity results are tight for a finite number of transmit and receive antennas and are accurate in the large-array limit.

Appendix

Proof of the Theorem. First we expand (5) in a more suitable form

$$\frac{N}{N}\frac{1}{N}\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{H}^{\mathrm{H}} = \left(\frac{\mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{d}}\mathbf{\Gamma}_{\mathrm{d}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}}\sigma_{\mathrm{R}}^2 + \sigma_{\mathrm{D}}^2\mathbf{I}_N}{N}\right)^{-1} \frac{\mathbf{\Theta}\mathbf{\Gamma}_{\mathrm{ud}}\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}^{\star}\mathbf{\Gamma}_{\mathrm{ud}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}}}{N^2}. \qquad (12)$$

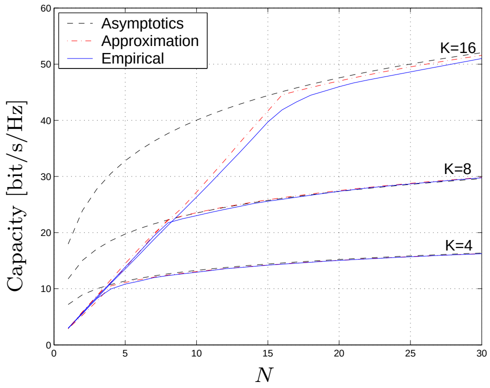

Fig. 4: Capacity of the relay-assisted MIMO Rayleigh channel vs. number of antennas N for $K = 4, 8, 16$ relays

For $k \neq l$ and $r_{k,l} = \frac{d}{\lambda}(\sin\phi_k - \sin\phi_l) \notin \mathbb{Z}$ we analyze the product

$$\begin{aligned}\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{\Phi}_l^{\star} &= \sum_{m=0}^{N-1} e^{j2\pi r_{k,l}m} = \sum_{m=0}^{N-1} z_{k,l}^m = \frac{1-z_{k,l}^N}{1-z_{k,l}} = \frac{|1+(-z_{k,l}^N)|}{|1-z_{k,l}|} \\ &\leq \frac{1+|z_{k,l}^N|}{|1-z_{k,l}|} = \frac{1+|z_{k,l}|^N}{|1-z_{k,l}|} = \frac{2}{|1-z_{k,l}|}, \end{aligned} \quad (13)$$

where we used the triangle inequality to bound the finite geometric series. For applicability of the bound (13) we have to demand that the source has to see every relay under a different angle (otherwise the eigenvectors belonging to aligned relays are spatially not separable). Under this separability assumption we have in the large-array limit $\frac{1}{N}\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{\Phi}_l^{\star} \stackrel{N\to\infty}{\longrightarrow} 0$.

For $k = l$ we have $r_{k,k} = 0$ and $\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{\Phi}_k^{\star} = N$. Thus we have for the array steering matrix

$$\frac{1}{N}\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}^{\star} \stackrel{N\to\infty}{\longrightarrow} \mathbf{I}_K.$$

The same considerations hold for the receive array response matrix

$$\frac{1}{N}\mathbf{\Theta}^{\mathrm{H}}\mathbf{\Theta} \stackrel{N\to\infty}{\longrightarrow} \mathbf{I}_K.$$

Using the property that for a non-singular matrix $\mathbf{X} \in \mathbb{C}^{N\times N}$ we have $\lambda_k(\mathbf{X}^{-1}) = \lambda_k^{-1}(\mathbf{X})$, we can write for $K < N$

$$\lambda_k\left(\left(\mathbf{\Theta\Gamma}_{\mathrm{d}}\mathbf{\Gamma}_{\mathrm{d}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}}\sigma_{\mathrm{R}}^2 + \sigma_{\mathrm{D}}^2\mathbf{I}_N\right)^{-1}\right) = \begin{cases} \lambda_k^{-1}(\mathbf{\Theta}^{\mathrm{H}}\mathbf{\Theta\Gamma}_{\mathrm{d}}\mathbf{\Gamma}_{\mathrm{d}}^{\mathrm{H}}\sigma_{\mathrm{R}}^2 + \sigma_{\mathrm{D}}^2\mathbf{I}_K) & ,k = 1\ldots K \\ 1/\sigma_{\mathrm{D}}^2 & ,k = K+1\ldots N \end{cases}$$

where we used $\lambda_k(c\mathbf{I}+\mathbf{X}) = c+\lambda_k(\mathbf{X})$, $\mathrm{rk}\left(\mathbf{\Theta\Gamma}_{\mathrm{d}}\mathbf{\Gamma}_{\mathrm{d}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}}\right) = K$ and $\lambda_k(\mathbf{AB}) = \lambda_k(\mathbf{BA})$ for $k = 1,\ldots,K$ and where $\mathbf{A} \in \mathbb{C}^{N\times K}$ and $\mathbf{B} \in \mathbb{C}^{K\times N}$. We obtain in the large-array limit for $k = 1,\ldots,K$

$$\lambda_k^{-1}\left(\frac{\mathbf{\Theta\Gamma}_{\mathrm{d}}\mathbf{\Gamma}_{\mathrm{d}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}}\sigma_{\mathrm{R}}^2 + \sigma_{\mathrm{D}}^2\mathbf{I}_N}{N}\right) \stackrel{N\to\infty}{\longrightarrow} \lambda_k^{-1}\left(\mathbf{\Gamma}_{\mathrm{d}}\mathbf{\Gamma}_{\mathrm{d}}^{\mathrm{H}}\sigma_{\mathrm{R}}^2\right) \quad (14)$$

and

$$\lambda_k\left(\frac{1}{N^2}\mathbf{\Theta\Gamma}_{\mathrm{ud}}\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}^{\star}\mathbf{\Gamma}_{\mathrm{ud}}^{\mathrm{H}}\mathbf{\Theta}^{\mathrm{H}}\right) \stackrel{N\to\infty}{\longrightarrow} \lambda_k\left(\mathbf{\Gamma}_{\mathrm{ud}}\mathbf{\Gamma}_{\mathrm{ud}}^{\mathrm{H}}\right). \quad (15)$$

The matrices obtained in (14) and (15) are diagonal and hence, the asymptotic eigenvalues are the diagonal elements

$$\lambda_k\left(\frac{1}{N}\mathbf{R}_{nn}^{-1}\mathbf{HH}^{\mathrm{H}}\right) \stackrel{N\to\infty}{\longrightarrow} \frac{|h_{\mathrm{u},k}g_k h_{\mathrm{d},k}|^2}{\sigma_{\mathrm{R}}^2|g_k h_{\mathrm{d},k}|^2} = \frac{|h_{\mathrm{u},k}|^2}{\sigma_{\mathrm{R}}^2} \quad (16)$$

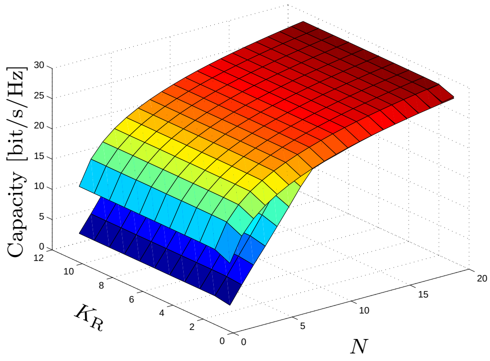

Fig. 5: Capacity of the relay-assisted MIMO Rice channel vs. Rice factor K_{R} and number of antennas N for $K = 8$ relays

for $k = 1,\ldots,K$ when $g_k \neq 0$ and $h_{\mathrm{d},k} \neq 0$ (this is satisfied with probability 1 for the channel model given in (8)). Obviously, the number of nonzero eigenvalues cannot exceed K since in the asymptotic case we have $\mathrm{rk}(\mathbf{R}_{nn}^{-1}\mathbf{HH}^{\mathrm{H}}) = \min\{K, N\} = K$, so that $\lambda_k = 0$ for $k = K+1\ldots N$.

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