# ON THE PERFORMANCE OF UWB GEO-REGIONING

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## ABSTRACT

Geo-regioning is an approach to localize Ultra-Wideband (UWB) transmitters by means of their channel impulse responses (CIRs) exploiting the capability of high temporal resolution of multipath components due to the large system bandwidth. Recently, the principle feasibility of this method was shown and a first geo-regioning algorithm was presented in [1, 2]. In this paper, this algorithm is analyzed and the achievable performance in terms of pairwise error probability (PEP) is computed. In order to calculate the PEP analytically, a simple and accurate approximation of the probability density function (PDF) of a weighted sum of squared real valued Gaussian random variables (RVs) is presented. In the course of this analysis a distance measure is found, which reveals the important characteristics of individual geo-regions determining the PEP.

## 1. INTRODUCTION

One of the most cited advantages of UWB technology is the capability of performing accurate localization [3]. The huge bandwidth introduces very high temporal multipath resolution to the propagation channel including an accurate representation of the initial delay. Therefore, most localization and ranging approaches in UWB are based on *Time-of-Arrival* (ToA) or *Time-Difference-of-Arrival* (TDoA) estimation [4].

In [1, 2], the authors present a different approach exploiting the nature of UWB channels to achieve a rough localization. They supposed that the CIR of a transmitter/receiver (TX/RX) pair is almost unique, due to many resolvable multipath components that result from the individual geographical constellation of RX and TX. At a given RX position the CIR received from a TX acts like a signature of its position. If some TXs have similar signatures, they are very likely located in the same region. We refer to the clustering of these TXs into geographical regions as *geo-regioning*.

As a first step *data aided geo-regioning* was investigated, which means that the positions of specific reference nodes are known. Moreover, these nodes have some a-priori knowledge on the characteristics of the regions they want to distinguish. With this information, the corresponding regions of transmitting nodes can be derived by an appropriate regioning process. This facilitates a variety of location aware services and protocols in dense ad-hoc networks.

In [1], it was found that the regioning algorithm needs several hundred estimated CIR taps to achieve good performance, which is too much for applications. Thus, this paper focuses on the analysis of the proposed maximum likelihood (ML) decision algorithm and the computation of the PEP, in order to predict and enhance the achievable performance and understand the significant differences of two regions determining the PEP.

In the course of these investigations a thorough statistical analysis of the log-likelihood decision variable is performed, which results in an analytical PEP approximation. There exist several numerical methods to calculate the PEP [5], but there is no analytical approximation available in literature to the best of author's knowledge. Since we are interested in performance determining parameters, we are looking for a mathematically tractable expression, where the dependency of the PEP on these parameters is explicitly visible. This is accomplished by the analytical approximation of the PEP presented in this paper.

In order to assess the difference of two regions, a distance measure is introduced, which induces criterions on the representation of two regions such that they can be distinguished with a low PEP.

## 2. PROBLEM STATEMENT

The geo-regioning algorithm is based on the spatial average power delay profiles (APDPs) of different geographical regions. The APDP of region $A$ is determined by

$$\mathrm{APDP}_A(k) = \frac{1}{N_A}\sum_{m=1}^{N_A} |h_{A,m}(k)|^2, \qquad (1)$$

where $N_A$ is the number of available CIR measurements or signatures from region $A$ represented by the sampled series $h_{A,m}(k)$, where $k = 1 \ldots K$ is the time index and $K$ is the number of measured CIR taps. The $m$-th power delay profile (PDP) of region A is then $|h_{A,m}(k)|^2$. In order to generate the APDP, the individual PDPs must be aligned properly in time. Since we do not request any timing information for our algorithm, we align the measured PDPs on the strongest multipath component. The alignment procedure is considered more thoroughly in section 4.

For now it is assumed that the APDPs of all regions are a-priori known at the receiver. To derive a simple algorithm, which assigns a received signature to a region, following simplifying assumptions are made:

- Each CIR or signature $\tilde{x}(k)$ with $k = 1...K$ is an outcome of a random process that generates statistically independent, zero mean, real valued Gaussian RVs. The variance at time index $k$ depends on the region, where the signature originates from and is set to the corresponding APDP value according to $\sigma_A^2(k) := \text{APDP}_A(k)$.

- A noisy estimate $x(k) = \tilde{x}(k) + n(k)$ of the signature $\tilde{x}(k)$ is observed, where the noise component $n(k)$ is additive zero mean Gaussian with variance $\sigma_n^2$. Hence, the RV $X(k)$ originating from region $A$ at time index $k$ is distributed according to $p(x(k)|A) = \mathcal{N}(0, \sigma_{A'}^2(k)) = \frac{1}{\sqrt{2\pi}\sigma_{A'}(k)} \exp\left(-\frac{x^2(k)}{2\sigma_{A'}^2}\right)$, where $\sigma_{A'}^2(k) = \sigma_A^2(k) + \sigma_n^2$ is the resulting variance of $X(k)$.

  The signal-to-noise ratio (SNR) is then defined according to

$$\text{SNR} = \frac{1}{2\sigma_n^2} \sum_{k=1}^{K} \left(\sigma_A^2(k) + \sigma_B^2(k)\right), \qquad (2)$$

  where $A$ and $B$ are the two different regions under test.

The ML estimator deciding between two regions $A$ and $B$ is

$$p(\vec{x}|A) \underset{B}{\overset{A}{\gtrless}} p(\vec{x}|B)$$

$$\sum_{k=1}^{K} x^2(k) \cdot \frac{\sigma_{A'}^2(k) - \sigma_{B'}^2(k)}{\sigma_{A'}^2(k)\sigma_{B'}^2(k)} \underset{B}{\overset{A}{\gtrless}} \sum_{k=1}^{K} \ln \frac{\sigma_{A'}^2(k)}{\sigma_{B'}^2(k)} = \delta, \quad (3)$$

where $\vec{x} = [x(1), x(2), \dots, x(K)]$ is the received signature with variances $\sigma_X^2(k)$ (c.f. [1]).

Introducing the RV $Z$ as the log-likelihood decision variable and $\xi(k) = \text{sign}(\sigma_{A'}^2(k) - \sigma_{B'}^2(k))$, we get

$$Z = \sum_{k=1}^{K} \xi(k) Y^2(k) \underset{B}{\overset{A}{\gtrless}} \delta, \qquad (4)$$

where each $Y(k)$ is a zero mean Gaussian RV with variance

$$\sigma_Y^2(k) = \sigma_X^2(k) \frac{|\sigma_{A'}^2(k) - \sigma_{B'}^2(k)|}{\sigma_{A'}^2(k)\sigma_{B'}^2(k)}. \qquad (5)$$

Consequently, the RV $Y^2(k)$ is distributed according to a Gamma distribution with parameters $\rho = 0.5$ and $\beta(k) = 2\sigma_Y^2(k)$. The Gamma PDF with the two parameters $\rho$ and $\beta$ is given by

$$G(\rho, \beta) = \frac{z^{\rho-1}\exp(-z/\beta)}{\Gamma(\rho)\beta^{\rho}} \quad z \geq 0 \text{ and } \rho, \beta > 0 \quad (6)$$

with the characteristic function (CF) $\Phi(s) = (1 - s\beta)^{-\rho}$.

### 2.1. Pairwise Error Probability

For the PEP calculation, it is assumed that the signature $\vec{x}$ originates either from region $A$ denoted as $\vec{x} \in A$ or from region $B$ denoted as $\vec{x} \in B$, which gives following variances for $Y(k)$:

$$\sigma_{Y_A}^2(k) = (\sigma_Y^2(k)|\vec{x} \in A) = \frac{|\sigma_{A'}^2(k) - \sigma_{B'}^2(k)|}{\sigma_{B'}^2(k)}$$

$$\sigma_{Y_B}^2(k) = (\sigma_Y^2(k)|\vec{x} \in B) = \frac{|\sigma_{A'}^2(k) - \sigma_{B'}^2(k)|}{\sigma_{A'}^2(k)}. \quad (7)$$

The PEP is defined as $P_{2e} = 0.5(P(Z \leq \delta|\vec{x} \in A) + P(Z > \delta|\vec{x} \in B)) = 0.5(P(Z \leq \delta|\vec{x} \in A) + 1 - P(Z \leq \delta|\vec{x} \in B))$. Thus, the probabilities $P(Z \leq \delta|\vec{x} \in A)$ and $P(Z \leq \delta|\vec{x} \in B)$ must be computed.

### 2.2. Exact Probability Computation

The exact probability $P(Z \leq \delta)$ is given by the inverse Laplace formula according to

$$P(Z \leq \delta) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\Phi_Z(s)}{s} \mathrm{e}^{(s\delta)} \mathrm{d}s, \qquad (8)$$

where $\Phi_Z(s)$ is the CF of $Z$. If the residues of $\frac{\Phi(s)}{s}\mathrm{e}^{(s\delta)}$ can be calculated either analytically or numerically, it is possible to evaluate this complex contour integral. In [5] several numerical methods like saddle-point integration or numerical integration by Gauss quadrature rules are described to compute this integral.

In this work an alternate approach is introduced to calculate $P(Z \leq \delta)$, which doesn't rely on the numerical computation of the inverse Laplace formula. The advantages of the proposed method are its simplicity next to producing accurate estimates.

## 3. APPROXIMATION OF THE PEP

### 3.1. Statistical Analysis of $Z$

The RV $Z$ is split into $K_1$ additive components ($\xi(k) = 1$) and $K - K_1$ subtractive components ($\xi(k) = -1$) according to

$$Z = Z_1 - Z_2 = \sum_{k=1}^{K_1} Y^2(k) - \sum_{k=K_1+1}^{K} Y^2(k), \quad (9)$$

where $Z_1$ and $Z_2$ are sums of independent gamma distributed RVs with constant $\rho$-parameters of 0.5 and varying $\beta$ parameters. The Laplace transform of a PDF is known as the CF and for each $Y^2(k)$ given by

$$\Phi_{Y^2(k)}(s) = \int_{-\infty}^{\infty} f^G_{Y^2(k)}(\tau)\mathrm{e}^{s\tau}\mathrm{d}\tau = (1 - s\beta(k))^{-0.5}. \quad (10)$$

Due to independency, the CFs for $Z_1$ and $Z_2$ are given by

$$\begin{aligned} \Phi_{Z_1}(s) &= \textstyle\prod_{k=1}^{K_1} (1 - s\beta(k))^{-0.5} \\ \Phi_{Z_2}(s) &= \textstyle\prod_{k=K_1+1}^{K} (1 - s\beta(k))^{-0.5} \end{aligned} \quad (11)$$

Hereafter, the two RVs $Z_1$ and $Z_2$ are written as $Z_q$ for $q \in \{1, 2\}$. The following considerations hold for both RVs.

In order to calculate the $n$-th order cumulants of the RV $Z_q$ the $n$-th derivative of the second characteristic functions $\Psi_{Z_q}(s) = \ln(\Phi_{Z_q}(s))$ must be evaluated for $s = 0$. Thus, the cumulants $c_{n,q}$ of $Z_q$ can be expressed in terms of the variances $\sigma^2_Y(k)$ (c.f. (5)) according to

$$\begin{aligned} c_{n,1} &= 2^{(n-1)} \cdot (n-1)! \textstyle\sum_{k=1}^{K_1} (\sigma^2_Y(k))^n \\ c_{n,2} &= 2^{(n-1)} \cdot (n-1)! \textstyle\sum_{k=K_1+1}^{K_2} (\sigma^2_Y(k))^n \end{aligned} \quad (12)$$

for $Z_1$ and $Z_2$, respectively.

It is not possible to find a mathematically tractable analytical expression for the PDFs of the RVs $Z_q$, which are the inverse Laplace transforms of the characteristic functions $\Phi_{Z_q}(s)$ according to (8).

The parameters for approximating Gaussian (c.f. 3.2) or Gamma (c.f. 3.3) PDFs are determined through the cumulants in (12).

### 3.2. Gauss Approximation

The first order cumulants (means) of $Z_q$ are subtracted to get the mean of $Z$ and the second order cumulants (variances) are added to get the variance of $Z$. With these parameters the equivalent Gaussian distribution (Gauss Approximation) is completely determined.

In order to calculate the PEP, two cases must be considered (c.f. subsection 2.1). If $\vec{x} \in A$, the mean and the variance of the Gauss Approximation are given by

$$\begin{aligned} \mu_A &= \textstyle\sum_{k=1}^{K} \xi(k)\sigma^2_{Y_A}(k) \\ \sigma^2_A &= 2\textstyle\sum_{k=1}^{K} (\sigma^2_{Y_A}(k))^2. \end{aligned} \quad (13)$$

If $\vec{x} \in B$, the corresponding mean and variance are calculated analogously. Thus, the approximated PEP is given by

$$\hat{P}_{2e} = \frac{1}{4} \cdot \left(2 + \mathrm{erf}\left(\frac{\delta - \mu_A}{\sqrt{2} \cdot \sigma_A}\right) - \mathrm{erf}\left(\frac{\delta - \mu_B}{\sqrt{2} \cdot \sigma_B}\right)\right),$$

where $\delta$ is defined in (4). This approximation, which can also be found by applying the central limit theorem, is due to big deviations for medium and high SNR values not applicable for our concerns and only mentioned for comparison. This means that there are too few RVs in the sum such that the central limit theorem is applicable.

### 3.3. Gamma Approximation

Since the summation of Gamma distributed RVs with identical $\beta$ parameters is again a Gamma distributed RV, it is natural to use a Gamma distribution to approximate the PDFs of the RVs $Z_q$.

The approximation procedure for the PDF of $Z$ is done in two steps. First the two RVs $Z_q$ are approximated separately by two Gamma distributions. Then these two Gamma PDFs are convolved, in order to calculate the PEP.

The parameters $\rho_q$ and $\beta_q$ of the equivalent Gamma PDFs are chosen such that means and variances of their PDFs correspond to the first two cumulants $c_{1,q}$ and $c_{2,q}$ (c.f. 12) according to

$$\begin{aligned} \rho_q &= \frac{(c_{1,q})^2}{c_{2,q}} \\ \beta_q &= \frac{c_{2,q}}{c_{1,q}}. \end{aligned} \quad (14)$$

The two approximating Gamma PDFs $f^G_{Z_1}(z_1)$ and $f^G_{Z_2}(z_2)$ (c.f. 6) are completely determined by these parameters.

Since the RVs $Z_1$ and $Z_2$ are assumed to be independent, the joint PDF of the pair $(Z_1, Z_2)$ can be expressed by the product of their marginal Gamma PDFs according to $f^G_{Z_1,Z_2}(z_1, z_2) = f^G_{Z_1}(z_1)f^G_{Z_2}(z_2)$. Therefore, the probability $P(Z = Z_1 - Z_2 \leq \delta)$ is the area under this joint PDF left from the line $z_1 - z_2 = \delta$ given by

$$\int_{z_2=-\infty}^{\infty} \int_{z_1=-\infty}^{\delta+z_2} f^G_{Z_1}(z_1)f^G_{Z_2}(z_2)\mathrm{d}z_1\mathrm{d}z_2$$

and visualized in Fig. 1. These considerations can be found in more detail in [6].

One simple way to evaluate this double integral numerically and calculate the probability $P(Z = Z_1 - Z_2 \leq \delta)$ is for $\delta > 0$ given by

$$P(0 \leq z_1 \leq \delta; 0 \leq z_2 < \infty) + \sum_{i=1}^{I} P(\delta + (i-1)\Delta \leq z_1 \leq \delta + i\Delta; i\Delta \leq z_2 < \infty)$$

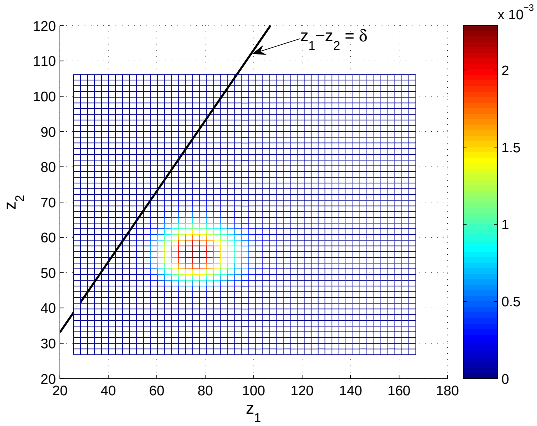


**Fig. 1**. Joint Gamma PDF

and for $\delta \leq 0$ by

$$\sum_{i=1}^{I} P((i-1)\varDelta \leq z_1 \leq i\varDelta; -\delta + i\varDelta \leq z_2 < \infty),$$

where the step size $\varDelta$ and the summation bound $I$ define the number of supporting points. For $\varDelta \rightarrow 0$ and $I \rightarrow \infty$, the sum approaches the searched probability.

## 4. DISTANCE MEASURE

In this section a simple measure for the difference of two given regions (APDPs) $A$ and $B$ is introduced, which is based on the analytical investigations from above and related to the PEP.

The decrease of the PEP for two given Gaussian PDFs with fixed variances and threshold is achieved by increasing the distance of the corresponding means, i.e., by shifting the two PDFs further away from each other.

This fact motivates a distance measure denoted as Distance of Means (DoM) given by

$$\begin{aligned} \text{DoM} &= \mu_A - \mu_B = \textstyle\sum_{k=1}^{K} \left( \frac{\sigma_A^2(k)}{\sigma_B^2(k)} + \frac{\sigma_B^2(k)}{\sigma_A^2(k)} - 2 \right) \\ &= \textstyle\sum_{k=1}^{K} \left( \alpha(k) + \frac{1}{\alpha(k)} - 2 \right), \end{aligned} \tag{15}$$

where $\alpha(k)$ is the ratio of the APDPs at time index $k$ and $\mu_A$ and $\mu_B$ are the means of the two PDFs of the RVs $(Z|\vec{x} \in A)$ and $(Z|\vec{x} \in B)$. Consequently, increasing the DoM, which is achieved by high $(\alpha(k) >> 1)$ or low $(\alpha(k) << 1)$ APDP-ratios, should decreases the PEP and thus increase the reliability in distinguishing the two regions.

However, the statistical investigations show that by changing the means, i.e., increasing the DoM, the variances are also effected, according to equation (12). Thus, the assumptions of a fixed variance and threshold do not hold. Nevertheless, simulation results (c.f. Fig. 3) show that the DoM can be uses in many cases as a rough distance measure.

One application of the DoM is to use it for the generation of the APDPs (c.f. (1)). As mentioned we do not request ToA information. Hence, the received PDPs must be aligned properly to generate an APDP. This can be done by aligning the PDPs to the peak value, i.e., to the arrival time of the strongest path. A problem occurs if there exist more than one sample within a PDP with approximately the same peak values but different time indices, such that the aligning algorithm doesn't know, on which sample to align. To avoid this, virtual APDPs with the different alignment possibilities are generated, the corresponding DoM values are computed and the alignment method achieving the maximum DoM value is chosen. The performance improvement of this method is depicted in Fig. 4.

## 5. SIMULATION RESULTS

### 5.1. Approximation Accuracy

In order to asses the accuracy of the introduced Gamma Approximation, Monte Carlo simulation results with $10^6$ samples, drawn from a normal distribution and processed according to the ML algorithm (c.f. (3)), are used as reference results. They are considered reliable for a PEP $> 10^{-4}$, which is the area of interest for the UWB geo-regioning approach.

The geographical regions are represented by 80-tap-APDPs $\sigma_A^2(k)$ and $\sigma_B^2(k)$, which are generated randomly according to a Gamma distribution with $\rho = 0.5$ and $\beta = 4$.

1000 simulation runs for SNRs (c.f. (2)) between 0 and 25 dB are performed. Quantile plots are shown in Fig. 2, which means that the 1000 simulation runs are sorted according to ascending relative approximation error ($\frac{P_{2e} - \hat{P}_{2e}}{P_{2e}}$) at an SNR of 22 dB and the 100th (10 percent quantile) and 900th (90 percent quantile) simulation runs are plotted.

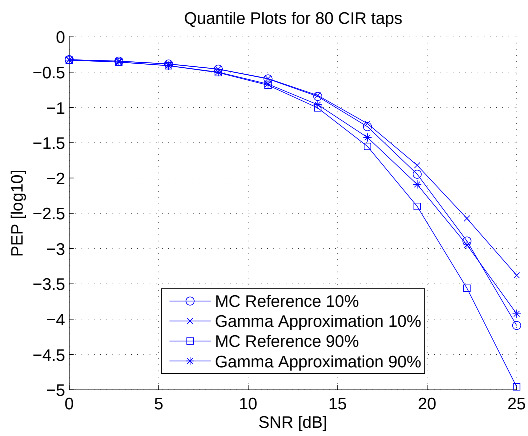


**Fig. 2**. PEP Approximation

### 5.2. Distance Measure

In order to evaluate the distance measure, 1000 simulation runs are performed, where the APDPs are generated as in section 5.1 and the SNR is set to 25 dB. For each run the PEP and the DoM are computed. The results are sorted according to descending PEP and visualized in Fig. 3.

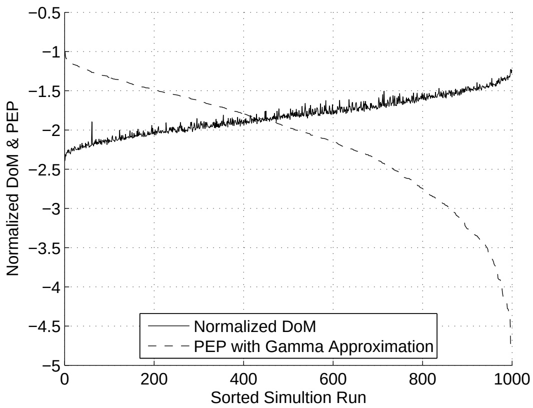


**Fig. 3**. Distance Measure

Fig. 3 shows that for decreasing PEP the DoM increases on average and can be used as distance measure. There is a little variance visible, which indicates that for some special cases the increasing of the DoM doesn't help to increase the performance.

As one of the possible applications of the DoM, Fig. 4 shows the achievable performance improvement, when the alignment of the PDPs to generate an APDP is done according to DoM algorithm described in section 4. These results are obtained from measured PDPs described in [7].

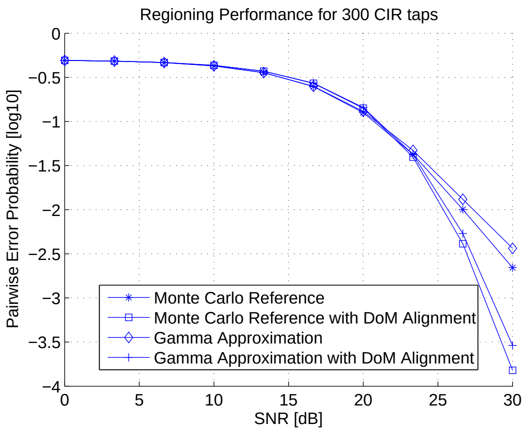


**Fig. 4**. Performance of DoM alignment

### 6. CONCLUSIONS

In this paper, we have introduced a simple method of approximating the PDF of the sum of weighted, squared Gaussian RVs with an equivalent Gamma PDF. Simulation results have shown that the approximation is accurate enough to use it for PEP computation of the proposed geo-regioning algorithm. This simple approximation can be used whenever the PDF of a sum of weighted, squared Gaussian RVs is searched. With this approximation it is possible to evaluate the geo-regioning algorithm instantaneously and get an insight into performance bounds depending on system parameters such as bandwidth, region size, number of CIR taps, number of antennas, propagation environment, etc.

Additionally, a simple distance measure (DoM) has been found, which indicates that two APDPs with very high or low ratios ($\alpha(k) >> 1$ or $\alpha(k) << 1$) can be distinguished very well. It was shown that the alignment procedure can be improved by using the DoM. A direct connection between the APDPs and the achievable performance has been established. This is a major breakthrough for further investigations, since the influence of the system parameters mentioned above on the achievable performance is henceforth transparent.

### 7. REFERENCES


[1] F. Althaus, F. Troesch, and A. Wittneben, "UWB geo regioning in rich multipath environment," in *IEEE Vehicular Technology Conference, VTC Fall 2005*, September 2005.

[2] ——, "Geo regioning in UWB networks," in *IST Mobile Summit, Dresden 2005*, June 2005.

[3] D. Porcino and W. Hirt, "Ultra-wideband radio technology: Potential and challenges ahead," *IEEE Communications Magazine*, vol. 41, no. 7, pp. 66–74, July 2003.

[4] S. Gezici, Z. Tian, G. Giannakis, H. Kobayashi, A. Molisch, H. Poor, and Z. Sahinoglu, "Localization via ultra-wideband radios," *IEEE Signal Processing Magazine*, pp. 70–84, July 2005.

[5] E. Biglieri, G. Caire, and G. Taricco, "Approximating the pairwise error probability for fading channels," *Electronics Letters*, vol. 31, no. 19, pp. 1625–1627, Sep 1995.

[6] A. Papoulis and S. U. Pillai, *Probability, Random Variables, and Stochastic Processes*, fourth edition ed. McGraw-Hill, 2002.

[7] F. Althaus, F. Troesch, T. Zasowski, and A. Wittneben, "STS measurements and characterization," *PULSERS Deliverable D3b6a*, vol. IST-2001-32710 PULSERS, 2005.